Isolated singularities table:

- $f$ is analytic in $D(z_0, r) \setminus \{z_0\}$, some $r > 0$

<table>
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<tr>
<th>Type of isolated singularity</th>
<th>Laurent expansion definition</th>
<th>Characterization in terms of $\lim_{z \to z_0} f(z)$</th>
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<tr>
<td>removable</td>
<td>$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ (no negative powers in Laurent)</td>
<td>1. $\lim_{z \to z_0} f(z) \neq \pm \infty$, a finite #, or...</td>
</tr>
<tr>
<td>pole (North pole!)</td>
<td>$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=0}^{N} b_m (z - z_0)^m$ with $b_N \neq 0$</td>
<td>2. $</td>
</tr>
<tr>
<td>simple pole if $N=1$</td>
<td></td>
<td>3. $\lim_{z \to z_0} f(z) (z - z_0)^N = 0$</td>
</tr>
<tr>
<td>essential singularity</td>
<td></td>
<td>4. $\forall \varepsilon &gt; 0$, $f(z) \neq 0$ for all $z \in D(z_0, \varepsilon) \setminus {z_0}$</td>
</tr>
</tbody>
</table>

In fact, none is true, called Picard's Theorem:

- $f(D(z_0, r) \setminus \{z_0\})$ contains all of $\mathbb{C}$, except for at most one point.

- e.g., $f(z) = e^{1/z}$, $z_0 = 0$.

Today, we explain column 3 characterizations.
removable singularity: If \( f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \) (Laurent characterization) in \( D(z_0,r) \setminus \{z_0\} \), then since this power series also converges at \( z_0 \), it defines an analytic function in \( D(z_0,r) \).

\[ \Rightarrow (1) \lim_{z \to z_0} f(z) = f(z_0) = a_0 \quad \text{exists} \quad (\text{since analytic } \Rightarrow \text{ continuous}) \]

\[ \Rightarrow (2) f \text{ bounded near } z_0; \text{ in fact for } M = |a_0| + 1 \]

\[ \int_0^{2\pi} \text{s.t. } 0 < r < \delta \Rightarrow |f(z)| < M \]

\[ \Rightarrow (3) \lim_{z \to z_0} |f(z)(z-z_0)| = \lim_{z \to z_0} M|z-z_0| = 0 \]

so \( \lim_{z \to z_0} f(z)(z-z_0) = 0 \)

The circle is completed if we now (3) \( \Rightarrow \) Laurent characterization

\[ \Rightarrow \begin{align*}
\quad & \text{then } \eta(z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^m} \, dz
\quad \text{(Monday notes)}
\end{align*} \]

The pole definition:

\[ q(z) = (z-z_0)^m f(z) \]

has only non-negative terms in its Laurent expansion; so extends to be analytic at \( z_0 \) with \( q(z_0) = b_0 \neq 0 \)

\[ \Rightarrow (1) \quad \lim_{z \to z_0} |f(z)|
\]

\[ = \lim_{z \to z_0} \frac{1}{(z-z_0)^m} |q(z)|
\]

\[ \Rightarrow \text{max } \text{ at } \infty \quad b_0 \neq 0 \]

So it remains to show

\[ (1) \Rightarrow \text{Laurent def of pole.} \]
\( L = \) Lament def of pole:

\[
\lim_{z \to z_0} f(z) = \infty.
\]

\[
(\text{at } k(z) = \frac{1}{f(z)})
\]

\[
\lim_{z \to z_0} k(z) = 0 \quad \Rightarrow \quad k(z) \text{ has a removable singularity at } z_0
\]

\[
\Rightarrow \quad k(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n, \quad c_n \neq 0, \quad n > 0 \quad \text{(since } k(z_0) = 0) \]

\[
k(z) = (z-z_0)^N h(z) \quad h(z_0) \neq 0
\]

\[
f(z) = \frac{1}{k(z)} = \frac{1}{(z-z_0)^N h(z)}
\]

\[
\text{analytic near } z_0 \text{, so has Taylor series}
\]

\[
\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad a_0 \neq 0
\]

i.e.

\[
f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad a_0 \neq 0
\]

\[\text{essential singularity:}\]

By logic (1), it suffices to show that if it is not true that

\[
\forall \epsilon > 0, \quad f(D(z_0, \epsilon) \setminus \{z_0\}) = \emptyset
\]

then \( z_0 \) is either a pole or a removable singularity !!

So, assume

\[
\exists \epsilon > 0 \text{ with } \overline{f(D(z_0, \epsilon) \setminus \{z_0\})} \neq \emptyset
\]

i.e. \( \exists w_0, \epsilon \text{ s.t. } D(w_0, \epsilon) \cap \overline{f(D(z_0, \epsilon) \setminus \{z_0\})} = \emptyset \).

Define

\[
k(z): = \frac{1}{f(z) - w_0}
\]

\[
|k(z)| \leq \frac{1}{\epsilon} \quad \forall z \in D(z_0, \epsilon) \setminus \{z_0\}
\]

\[
\Rightarrow k \text{ has removable sing } \text{ at } z_0
\]

\[
k(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad N > 0, \quad a_N \neq 0
\]

\[
\frac{1}{f(z) - w_0} = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad a_0 \neq 0
\]

\[
= (z-z_0)^N h(z) \quad h(z_0) \neq 0
\]

\[
f(z) - w_0 = \frac{1}{(z-z_0)^N h(z)}
\]

\[
f(z) = w_0 + \frac{1}{(z-z_0)^N h(z)}, \quad h(z_0) \neq 0 \quad \text{i.e. } f \text{ has a removable singularity or a pole at } z_0
Justification for computing Laurent series coefficients of a product, by term by term multiplication:

**Theorem** Let \( f(z) \), \( g(z) \) have Laurent series \( \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \) in \( A = \left\{ z \mid r_1 < |z-z_0| < r_2 \right\} \)

\[
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{c_m}{(z-z_0)^m} = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n
\]

\[
g(z) = \sum_{k=-\infty}^{\infty} b_k (z-z_0)^k
\]

Then \( f(z)g(z) \) has Laurent Series

\[
f(z)g(z) = \sum_{n=-\infty}^{\infty} d_n (z-z_0)^n
\]

where

\[
d_n = \lim_{M,N \to \infty} \sum_{j=-M}^{N} a_j b_{n-j} = \sum_{j=-\infty}^{\infty} a_j b_{n-j}
\]

**Proof:** Recall we recover \( d_n \) with a contour integral: for \( r_1 < r < r_2 \)

Fix \( n \in \mathbb{Z} \).

\[
d_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)g(z)}{(z-z_0)^{n+1}} \, dz
\]

\[
(f \ast f_{M,N})(z) = \sum_{j=-M}^{N} a_j (z-z_0)^j, \quad (g \ast g_{M,N})(z) = \sum_{k=-M}^{N} b_k (z-z_0)^k
\]

\[
f_{M,N} \to f \text{ uniformly, as } M,N \to \infty, \text{ on } \gamma = \{ z \mid |z-z_0| = r \}
\]

\[
g_{M,N} \to g \text{ uniformly on } \gamma
\]

\[
\Rightarrow f_{M,N}g_{M,N} \to fg \text{ uniformly on } \gamma
\]

\[
\Rightarrow (\text{via } \ast), \text{ that } d_n \text{ is the limit as } M,N \to \infty, \text{ of the } n^{\text{th}} \text{ Laurent coefficient } d_n \text{ of } f \ast g_{M,N}. \text{ But this product is a product of finite sums, and the unique sum of } 2^n \text{ in its Laurent series is precisely the finite sum obtained by term by term multiplication:}
\]

\[
\sum_{j=-M}^{N} a_j b_{n-j}
\]
The wonderful § 4.2

Residue Theorem \( \text{Let } f: \mathbb{C} \setminus \{z_1, z_2, \ldots, z_N\} \text{ be analytic and simply connected.} \)

(\text{Let } \gamma \text{ be a p.w. } C^1 \text{ closed curve in } \mathbb{C} \text{ (i.e., } \gamma: [a,b] \to \mathbb{C} \).

Then \( \int_{\gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{N} \text{Res}(f; z_k) \, I(\gamma; z_k) \). \)

\text{Proof: at each } z_k \text{, } f \text{ has Laurent expansion}

\[ f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \uparrow \text{ convergent in } \mathbb{C} \setminus \{z_k\} \]

Thus \( f(z) - \sum_{k=1}^{N} S_2(z) \) has removable singularities at each \( z_k \), and so can be thought of as analytic in \( \mathbb{C} \).

Thus \( \int_{\gamma} f(z) \, dz - \sum_{k=1}^{N} S_2(z) \, dz = 0 \text{ by deformation theorem.} \)

\( \therefore \int_{\gamma} f(z) \, dz = \sum_{k=1}^{N} \int_{\gamma} S_2(z) \, dz = \sum_{k=1}^{N} \int_{\gamma} \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m} \, dz \)

\text{Interchange } \sum, \int \text{ by unif. cont.}

\[ = \sum_{k=1}^{N} \sum_{m=1}^{\infty} b_m \, \text{Im}(I(\gamma; z_k)) \]

\text{Example:} \( \int_{|z|=2} \frac{e^z}{z^2 - 1} \, dz \)