Math 4200-1
Fri Nov. 9

HW for Fri Nov 16
4.2 2, 3, 4, 6, 8, 13, 15
4.3 1, 2, 3, 9, 10, 14, 17, 20b

NOTE $\longrightarrow$ Exam 2 will be Monday Nov. 19, and will cover thru §4.3

§3.3, 4.2

Isolated singularities table:

f is analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$

Type of isolated singularity	Laurent expansion definition	characterization in terms of $\lim_{z\to z_0} f(z)$
removable (because f extends to be analytic in $D(z_0;r)$)	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in Laurent)	① $\lim_{z\to z_0} f(z)$ $\exists$, as a finite #, or ② $\lvert f(z)\rvert\le M$ $\forall$ $\lvert z-z_0\rvert\le\rho$, some $0<\rho<r$, or ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$
pole (North pole!) of order N simple pole if N=1	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{N}\frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$	① $\lim_{z\to z_0} f(z)=\infty$, (the North pole!) or ② $\exists$ N s.t. $g(z):=(z-z_0)^N f(z)$ has a removable singularity @ $z=z_0$, with $g(z_0)\neq 0$
essential singularity	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}$ s.t. $\exists$ $m_j\to\infty$ s.t. $b_{m_j}\neq 0$	$\forall$ $0<\rho<r$, $\overline{f(D(z_0,\rho)\setminus\{z_0\})}=\mathbb{C}$ [In fact, more is true, called Picard's Theorem: $f(D(z_0,\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$, except for at most one point - e.g. $f(z)=e^{1/z}$, $z_0=0$].

Today, we explain column 3 characterizations.

removable singularity : If $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$ (Laurent characterization)

in $D(z_0,r)\setminus\{z_0\}$

then since this power series also converges at z_0,
it defines an analytic function in $D(z_0,r)$

$\Rightarrow$ ① $\lim_{z\to z_0} f(z) = f(z_0) = a_0$ exists (since analytic $\Rightarrow$ continuous)

$\Rightarrow$ ② f bounded near z_0; in fact for $M=|a_0|+1$
$\exists\, \delta>0$ s.t. $0<|z-z_0|<\delta \Rightarrow |f(z)|\le M$

$\Rightarrow$ ③ $\lim_{z\to z_0} |f(z)(z-z_0)| \le \lim_{z\to z_0} M|z-z_0| = 0$

so $\lim_{z\to z_0} f(z)(z-z_0)=0$

The circle is completed if we show ③ $\Rightarrow$ Laurent characterization

Take γ: $\zeta(t)=\rho e^{it}$ $0<\rho<r$, $0\le t\le 2\pi$

then $m\ge 1$ $b_m = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta-z_0)^{m-1}\,d\zeta$ (Monday notes)

③ $\Rightarrow b_m=0\ \forall m$:

$$|b_m| \le \frac{1}{2\pi}\int_\gamma |f(\zeta)|\,\rho^{m-1}\,|d\zeta|$$

$$\le \underbrace{\frac{1}{2\pi}\rho^{m-1}2\pi\rho}_{\substack{\to 0\ (m>1)\\ \text{cont}\ (m=1)}} \underbrace{\max\{|f(\zeta)|,\ |\zeta-z_0|=\rho\}}_{\substack{\to 0 \text{ as } \rho\to 0\\ \text{by ③}}}$$

pole

Laurent definition

$\Rightarrow$ $g(z) := (z-z_0)^N f(z)$

has only non-negative terms in its Laurent expansion, so extends to be analytic at z_0, with

② $g(z_0) = b_N \ne 0$

② $\Rightarrow$ ① since $\lim_{z\to z_0} |f(z)|$

$= \lim_{z\to z_0} \frac{1}{|z-z_0|^N}\,|g(z)|$

$\frac{1}{|z-z_0|^N}\to\infty$, $|g(z)|\to |g(z_0)|\ne 0$

So it remains to show
① $\Rightarrow$ Laurent def of pole.

1 $\Rightarrow$ Laurent def of pole:

$$\lim_{z \to z_0} f(z) = \infty.$$

let $k(z) = \frac{1}{f(z)}$

$\Rightarrow \lim_{z \to z_0} k(z) = 0 \Rightarrow k(z)$ has a removable singularity at z_0

$\Rightarrow k(z) = \sum_{n=N}^{\infty} c_n (z-z_0)^n \qquad c_N \neq 0,\ N \geq 1$ (since $k(z_0)=0$)

in some $D(z_0, \rho)$

$k(z) = (z-z_0)^N h(z) \qquad h(z_0) \neq 0$

$\Rightarrow f(z) = \frac{1}{k(z)} = \frac{1}{(z-z_0)^N} \underbrace{\frac{1}{h(z)}}$

analytic near z_0, so has Taylor series

$\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_0 \neq 0$

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n-N} \qquad a_0 \neq 0$ ∎

<u>essential singularity</u>:

By logic (!), it suffices to show that if it is <u>not</u> true that

$$\forall\, 0<\rho<r, \quad \overline{f(D(z_0;r) \setminus \{z_0\})} = \mathbb{C}$$

then z_0 is either a pole or a removable singularity !!

So, assume

$$\exists\, 0<\rho<r \text{ with } \overline{f(D(z_0;\rho) \setminus \{z_0\})} \neq \mathbb{C}$$

i.e. $\exists\ w_0, \varepsilon$ s.t. $D(w_0;\varepsilon) \cap \overline{f(D(z_0;\rho) \setminus \{z_0\})} = \emptyset$.

Define $k(z) := \frac{1}{f(z)-w_0}$

$|k(z)| \leq \frac{1}{\varepsilon} \qquad \forall z \in D(z_0,\rho) \setminus \{z_0\}$

$\Rightarrow k$ has removable sing @ z_0

$k(z) = \sum_{n=N}^{\infty} a_n (z-z_0)^n \qquad N \geq 0,\ a_N \neq 0$

$\frac{1}{f(z)-w_0} = \sum_{n=N}^{\infty} a_n (z-z_0)^n \qquad a_N \neq 0$

$= (z-z_0)^N h(z) \qquad h(z_0) \neq 0$

$f(z) - w_0 = \frac{1}{(z-z_0)^N} h(z)$

$f(z) = w_0 + \frac{1}{(z-z_0)^N} h(z), \quad h(z_0) \neq 0;\ N \geq 0$ i.e. f has a removable singularity or a pole at z_0 ∎

Justification for computing Laurent series coefficients of a product, by term by term multiplication:

<u>Theorem</u> Let $f(z)$, $g(z)$ have Laurent series in $A = \{z \mid r_1 < |z-z_0| < r_2\}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{c_m}{(z-z_0)^m} := \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$g(z) = \sum_{k=-\infty}^{\infty} b_k (z-z_0)^k$$

Then $f(z)g(z)$ has Laurent Series

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} d_n (z-z_0)^n$$

where $$d_n = \lim_{M,N\to\infty} \sum_{j=-M}^{N} a_j b_{n-j} := \sum_{j=-\infty}^{\infty} a_j b_{n-j}$$

<u>Proof</u>: Recall we recover d_n with a Contour integral; for $r_1 < r < r_2$

Fix $n \in \mathbb{Z}$.

$$* \qquad d_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(\zeta)g(\zeta)}{(\zeta-z_0)^{n+1}} d\zeta$$

Let $$f_{M,N}(z) = \sum_{j=-M}^{N} a_j (z-z_0)^j \quad , \quad g_{M,N}(z) = \sum_{k=-N+n}^{M+n} b_k (z-z_0)^k$$

$$\left.\begin{array}{l} f_{M,N} \to f \\ g_{M,N} \to g \end{array}\right\} \text{uniformly, as } M,N\to\infty, \text{ on } \gamma = \{z \mid |z-z_0| = r\}$$

$\Rightarrow f_{M,N}\, g_{M,N} \to fg$ uniformly on γ

$\Rightarrow$ (via $*$), that d_n is the limit as $M,N\to\infty$, of the n^{th} Laurent coeff of $f_{M,N}\, g_{M,N}$. But this product is a product of <u>finite</u> sums, and the (unique) coeff of z^n in its Laurent series is precisely the finite sum obtained by term by term multiplication:

$$\sum_{j=-M}^{N} a_j b_{n-j} \qquad \blacksquare$$

The wonderful §4.2

<u>Residue Theorem</u> Let $f : A \setminus \{z_1, z_2, \dots z_N\}$ be analytic
A simply connected.

Let γ be a p.w. C^1 closed curve in A (i.e. $\gamma : [a,b] \to A$).

Then $$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k)\, I(\gamma; z_k)$$

<u>proof</u>: at each z_k f has Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} a_n^{(k)} (z-z_0)^k + \underbrace{\sum_{m=1}^{\infty} \frac{b_m^{(k)}}{(z-z_0)^m}}_{\text{conv in } \mathbb{C} \setminus \{z_k\}} := S_2^{(k)}(z)$$

Thus $$f(z) - \sum_{k=1}^{N} S_2^{(k)}(z)$$ has removable singularities at each z_k, and so can be thought of as analytic in A.

Thus $$\int_\gamma f(z) - \sum_{k=1}^{N} S_2^{(k)}(z)\,dz = 0$$ by deformation theorem.

i.e. $$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} \int_\gamma S_2^{(k)}(z)\,dz = \sum_{k=1}^{N} \underbrace{\int_\gamma \sum_{m=1}^{\infty} \frac{b_m^{(k)}}{(z-z_0)^m}\,dz}_{\text{interchange } \int, \Sigma \text{ by unif. conv.}}$$

$$= \sum_{k=1}^{N} b_1^{(k)}\, 2\pi i\, I(\gamma; z_k)$$ ∎

<u>Example</u> $$\int_{|z|=2} \frac{e^z}{z^2-1}\,dz$$