Math 4200
Wednesday Nov 7

§3.3 continued. Laurent Series. → first do last Example Monday, which was Laurent for $f(z) = \frac{e^z}{\sin z}$
§4.1 finding residues

If f analytic in $D(z_0,r)\setminus\{z_0\}$,
then the singularity at z_0 is called <u>isolated</u>.
(zeroes of non-zero analytic fcns are automatically isolated. Not so for singularities)

$\boxed{\text{Neat table}}$

Name (type) of isolated singularity at z_0	Laurent expansion definition	characterization in terms of $\lim\limits_{z\to z_0} f(z)$
<u>removable</u> (because you can extend f to be analytic at z_0)	$f(z)=\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$	① $\lim\limits_{z\to z_0} f(z)\ \exists$ ← as a <u>finite</u> number, or ② $\lvert f(z)\rvert \le M$ for some $\rho>0$, $\forall\ 0<\lvert z-z_0\rvert<\rho$, or ③ $\lim\limits_{z\to z_0} f(z)(z-z_0) = 0$
<u>pole</u> (means North pole!) of order N <u>simple pole</u> (of order 1)	$f(z)=\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n + \sum\limits_{m=1}^{N} \frac{b_m}{(z-z_0)^m}$, $b_N\ne 0$ $f(z)=\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0}$	① $\lim\limits_{z\to z_0} f(z) = \infty$ ← the north pole!, or ② $\exists\ N\in\mathbb{N}$ s.t. $g(z) := f(z)(z-z_0)^N$ has a removable singularity at z_0, and $g(z_0)\ne 0$
<u>essential singularity</u>	$f(z)=\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n + \sum\limits_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$, $\exists\ m_j\to\infty$ s.t. $b_{m_j}\ne 0$	$\forall\ 0<\rho<r$, $f(D(z_0,\rho)\setminus\{z_0\}) = \mathbb{C}$! [In fact, a hard result, called Picard's Thm, says $f(D(z_0,\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$ except for at most <u>one</u> point!] example: $e^{\frac{1}{z}}$, $z_0=0$.

We'll explain column 3 later
(it contains some nice theorems)
But we need to talk some about finding residues, for §4.1 hw.

Useful fact for computing Laurent Series of products (helps some 3.3 Hw)

Theorem : Let $f(z) = g(z)h(z)$ where g, h are analytic in $A = \{z \mid r_1 < |z - z_0| < r_2\}$.

If $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m} := \sum_{j=-\infty}^{\infty} c_j (z-z_0)^j$$ (a shorter way to write the Laurent series)

Then each c_j can be determined by term by term mult. of $g(z), h(z)$ Laurent series. (If g & h only have poles you can recover proof from one for Taylor. General case harder – we'll do on Friday)

Exercise 1 Compute c_0 for $f(z) = \dfrac{e^{\frac{1}{z}}}{1-z}$, in $0 < |z| < 1$ Laurent expansion.

Residues, §4.1: The text has a fairly imposing table on page 250, for how to compute residues. Let us investigate!

yipes!

the residue at z_0, $\mathrm{Res}(g/h; z_0)$ is given by

$$\mathrm{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times \begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ matrix.

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

	Function	Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0) \neq 0$, $h(z_0) = 0 = h'(z_0)$, $h''(z_0) \neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0) \neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0) = 0, g'(z_0) \neq 0$, $h(z_0) = 0 = h'(z_0)$ $= h''(z_0), h'''(z_0) \neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z) = (z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z) = (z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0) \neq 0, h(z_0) = \ldots = h^{k-1}(z_0) = 0, h^k(z_0) \neq 0$	pole of order k	see Proposition 4.1.7.

The fact is, despite the scary (and sometimes useful) table, residues ~~can~~ for

$$f(z) = \frac{g(z)}{h(z)}$$

with g, h analytic at z_0

can almost always be computed via the Taylor series of g & h, and clever use of geometric series.

Exercise 2 Verify entries 4, 6 in the table:

(4) If $f(z) = \frac{g(z)}{h(z)}$ $\quad$ $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$ (simple pole) $\quad$ then $\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}$

(6) If $f(z) = \frac{g(z)}{h(z)}$ $\quad$ $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$ (double pole) $\quad$ then $\operatorname{Res}(f; z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)\,h'''(z_0)}{[h''(z_0)]^2}$

<u>Exercise 3</u> Use Exercise 2 and/or direct argument, to find

$\text{Res}\,(\pi \cot \pi z \,;\, k)$, $k \in \mathbb{N}$

$\text{Res}\left(\frac{e^z}{\sin^2 z} \,;\, 0\right)$

```
> series(exp(z),z=0,4);
```

$$1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+O(z^4)$$

```
> series(exp(z)/sin(z)^2,z=0,4);
```

$$z^{-2}+z^{-1}+\frac{5}{6}+\frac{1}{2}z+O(z^2)$$

```
> series(exp(z)/sin(z)^10,z=0,10);
#try computing this residue by hand!
```

$$z^{-10}+z^{-9}+\frac{13}{6}z^{-8}+\frac{11}{6}z^{-7}+\frac{167}{72}z^{-6}+\frac{623}{360}z^{-5}+\frac{25111}{15120}z^{-4}+\frac{16973}{15120}z^{-3}+\frac{21839}{24192}z^{-2}+\frac{40885}{72576}z^{-1}+O(z^0)$$

```
>
```