Math 4200
Monday Nov. 5

Laurent series : Theorem

Let $A = \{z \mid r_1 < |z-z_0| < r_2\}$ be an open annulus. Then

(1) $f: A \to \mathbb{C}$ analytic,

$\boxed{\text{iff}}$

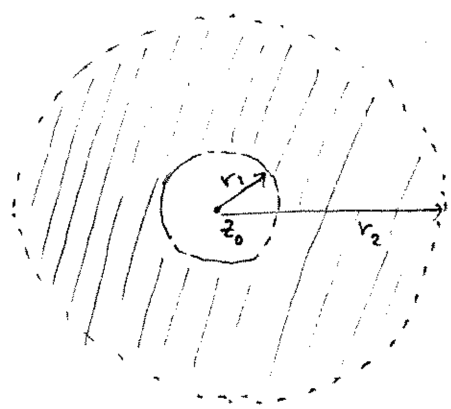
(2) $f(z)$ has a power series expansion (with positive & negative powers),

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$$

$$:= S_1(z) + S_2(z)$$

Here $S_1(z)$ converges for $|z-z_0| < r_2$ (uniformly absolutely for $|z-z_0| \le r_2 - \varepsilon$, $\varepsilon > 0$)
$S_2(z)$ converges for $|z-z_0| > r_1$ (uniformly absolutely for $|z-z_0| \ge r_1 + \varepsilon$, $\varepsilon > 0$)

i) Furthermore, the coefficients a_n, b_m are uniquely determined by f;
specifically, if $r_1 < r < r_2$,
and if $\gamma = \{z \mid |z-z_0| = r,\ r_1 < r < r_2\}$
is the radius r circle centered at z_0, then



$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta$$

(2) $\Rightarrow$ (1) : Since $S_1(z)$ and $S_2(z)$ converge uniformly on the sub-annuli $\{z \mid r_1 + \varepsilon < |z-z_0| < r_2 - \varepsilon\}$
their limits are analytic, and so $S_1(z) + S_2(z)$ is as well. ■

Note too that $S_1(z)$ converges for $|z-z_0| < r_2$ implies uniform absolute convergence
for $|z-z_0| \le r_2 - \varepsilon$; (We already had this discussion for power series.....
... conv. for $|z-z_0| = r_2 - \varepsilon/2 \Rightarrow M = \max\{|a_n|(r_2 - \varepsilon/2)^n,\ n = 0,1,2..\} < \infty$
then $|z-z_0| \le r_2 - \varepsilon \Rightarrow |a_n (z-z_0)^n| \le |a_n|(r_2-\varepsilon)^n \le M \left(\frac{r_2 - \varepsilon}{r_2 - \varepsilon/2}\right)^n = M\mu^n,\ \mu < 1$

$$\sum_{n=0}^{\infty} M\mu^n = \frac{M}{1-\mu} < \infty \quad ■$$

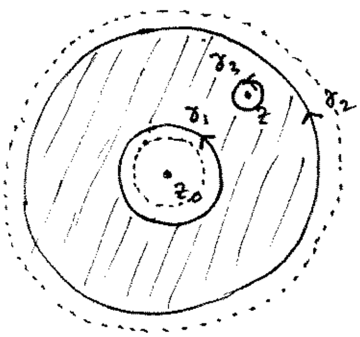
Similarly, $S_2(z)$ converges for $|z-z_0| > r_1$
implies uniform absolute convergence for $|z-z_0| \ge r_1 + \varepsilon$. See HW.

1 $\Rightarrow$ 2 : Let $\varepsilon > 0$.

Let $A_\varepsilon = \{ z \in A \mid r_1 + \varepsilon \le |z - z_0| \le r_2 - \varepsilon \}$

Let γ_1 = circle of radius $r_1 + \varepsilon/2$ about z_0
γ_2 = circle of radius $r_2 - \varepsilon/2$ about z_0
Let $z \in A_\varepsilon$, γ_3 = circle of radius $\varepsilon/4$ about z

} all circles oriented c.c., as usual

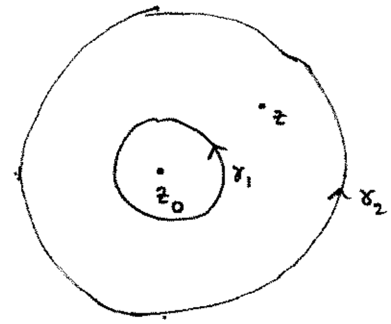


$\gamma_2 - \gamma_1 - \gamma_3$ is the oriented boundry of a region which is a disk with two holes, and this region is compactly contained within A.

Green's Thm version of Cauchy's formula

$$\Rightarrow \quad 0 = \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \underbrace{\int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{f(z)\, 2\pi i\, (\underbrace{I(\gamma_3; z)}_{1})}$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta}$$



$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = r_2 - \varepsilon/2} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$\frac{f(\zeta)}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}\, d\zeta$$

"w", $|w| \le \frac{1-\varepsilon}{1-\varepsilon/2} := \mu < 1$

$$\frac{1}{2\pi i} \int_{\gamma_2} \sum_{n=0}^{\infty} \underbrace{\frac{f(\zeta)}{(\zeta - z_0)} \left(\frac{z - z_0}{\zeta - z_0}\right)^n}\, d\zeta$$

$|\%| \le \frac{M}{r_2 - \varepsilon/2} \mu^n$

unif (abs) conv $\Rightarrow$ interchange of sum & integral is valid

$\|$

$$\underbrace{\sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta}_{:= a_n}}_{\| \atop S_1(z)}$$

$$+ \frac{1}{2\pi i} \int_{|\zeta - z_0| = r_1 + \varepsilon/2} \frac{f(\zeta)}{(z - z_0) - (\zeta - z_0)}\, d\zeta$$

$$\frac{f(\zeta)}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)}\, d\zeta$$

"w", $|w| \le \frac{r_1 + \varepsilon/2}{r_1 + \varepsilon} = \mu < 1$.

$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = r_1 + \varepsilon} \sum_{k=0}^{\infty} \frac{f(\zeta)}{z - z_0} \left(\frac{\zeta - z_0}{z - z_0}\right)^k d\zeta$$

Let $m = k+1$
$m = 1 \ldots \infty$,
interchange sum & integral

$\|$

$$\underbrace{\sum_{m=1}^{\infty} \frac{1}{(z - z_0)^m} \underbrace{\frac{1}{2\pi i} \int_{\gamma_1} f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta}_{:= b_m}}_{S_2(z)}$$

3 uniqueness:

First notice that for any $r_1 < r < r_2$

$$\int_{|\zeta - z_0| = r} f(\zeta)\,d\zeta = \int_{|\zeta - z_0| = r} S_1(\zeta)\,d\zeta + \int_{|\zeta - z_0| = r} S_2(\zeta)\,d\zeta .$$

Unif. conv $\Rightarrow \int \Sigma = \Sigma \int$, so

$$= \sum_{n=0}^{\infty} a_n \underbrace{\int_{|\zeta - z_0| = r} (\zeta - z_0)^n\,d\zeta}_{= 0 \text{ FTC.}} + \sum_{m=1}^{\infty} b_m \int_{|\zeta - z_0| = r} \frac{1}{(\zeta - z_0)^m}\,d\zeta$$

$$\int_{|\zeta - z_0| = r} \frac{1}{(\zeta - z_0)^m}\,d\zeta = \begin{cases} 0 & m = 2,3,\dots \text{ FTC!} \\ 2\pi i & m = 1 \end{cases}$$

so $\displaystyle\int_{|\zeta - z_0| = r} f(\zeta)\,d\zeta = 2\pi i\, b_1.$

For this reason, $\underline{b_1}$ is defined to be the <u>residue</u> of f (at z_0)
(because it's all that's left after you do the contour integral!)

But we can use this trick to pick off <u>any</u> coefficient in $S_1(z)$ or $S_2(z)$!

Let $n_1 \geq 0$. (integer of course.)

$$f(\zeta) = \sum_{n=0}^{\infty} a_n (\zeta - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(\zeta - z_0)^m}$$

$$\frac{f(\zeta)}{(\zeta - z_0)^{n_1+1}} = \sum_{n=0}^{\infty} a_n (\zeta - z_0)^{n-(n_1+1)} + \sum_{m=1}^{\infty} \frac{b_m}{(\zeta - z_0)^{m+n_1+1}}$$

Thus $\boxed{\displaystyle\int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n_1+1}}\,d\zeta = 2\pi i\, a_{n_1}}$ because all other terms integrate to zero.

Let $m_1 \geq 1$:

$$f(\zeta)(\zeta - z_0)^{m_1-1} = \sum_{n=0}^{\infty} a_n (\zeta - z_0)^{n+m_1-1} + \sum_{m=1}^{\infty} \frac{b_m}{(\zeta - z_0)^{m-m_1+1}}$$

$\Rightarrow \boxed{\displaystyle\int_{|\zeta - z_0| = r} f(\zeta)(\zeta - z_0)^{m_1-1}\,d\zeta = 2\pi i\, b_{m_1}}$ ∎

Examples

we did $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

$z_0 = 0$

in

(a) $|z|<1$ Taylor

(b) $1<|z|<2$ Laurent

(c) $|z|>2$ (well, we didn't do this one. But all 3 are done using geometric series)

Exercise 1 : Find the Laurent series for $ze^{\frac{1}{z}}$ in $\mathbb{C}\setminus\{0\}$,

and use it to deduce

$$\int_\gamma z e^{1/z}\, dz$$

if γ is a closed curve with $I(\gamma;0)=1$.

Exercise 2 : Let $f(z) = \dfrac{e^z}{\sin z}$, $0<|z|<\pi$.

Find the Laurent series, at least the first four non-zero terms.

Hint: First deduce

$z f(z)$

is analytic. Then use Taylors series thm about multiplying power series