Math 4200
Friday Nov. 30

HW for Friday Dec. 7
5.2 1, 4a, 6, 7, 10, 24, 26, 33, 34

Next week will be "presentation mode."
My lectures will be more "overview & survey", and also
preliminary material for Gary's presentation
We need to decide who among Gary, Cody, Jason + Christian will speak Wednesday
(the other two presentations will be Friday.)

Today: finish the portion of conformal transformations for which you will be responsible. (Applications Monday will be survey mode.)

We begin on page 2 Wed. notes, Exercise 4.

after Wed. notes:

Exercise 1: Find the inverse transformation to the one in Exercise 5.

Exercise 2 Map the half strip $\{z = x + iy \mid 0 < x < 1, y > 0\}$ conformally onto the unit disk, by exhibiting a sequence of transformations whose composition is the desired map. (You may need the back of this sheet!)

Exercise 3 Find all conformal diffeomorphisms of $D(0;1) \to D(0;1)$.

Hint: Find all FLT's which map $z_0 \in D(0;1)$ to zero, and the disk onto itself, then appeal to uniqueness part of R.M.T.

(These FLT's are called the Möbius transformations, and provide the missing degrees of freedom in our Monday examples.)

Hint 2: Do you recall an early HW exercise that $\left|\frac{z-w}{1-\bar{z}w}\right| = 1$ if $|z| = 1$ and $\bar{z}\bar{w} \neq 1$?

The Riemann sphere $\mathbb{C} \cup \{\infty\} = S^2_R$ as a complex manifold

Our atlas has 2 charts, the z-plane $\mathbb{C}_z$ & the w-plane $\mathbb{C}_w$

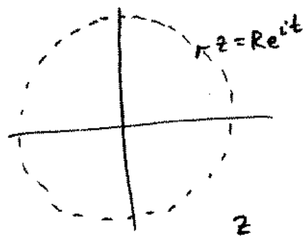
$$S^2_R := \frac{\mathbb{C}_z \cup \mathbb{C}_w}{\sim}$$

where the equivalence relation is that $z \sim \frac{1}{w}$ for $z, w \neq 0$.

$z = \infty$ now means $w = 0$
($w = \infty$ means $z = 0$)

Atlas page 1

$\mathbb{C}_z$



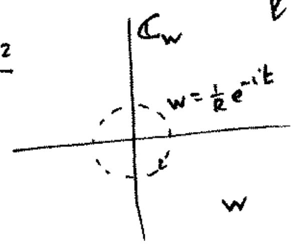
identification maps are analytic.

$w = \frac{1}{z}$ $(z \neq 0)$

$z = \frac{1}{w}$ $(w \neq 0)$

Atlas page 2

$\mathbb{C}_w$



(Think of charts of S^2 obtained by stereographic proj. from north & south poles; it's almost the same.)

Def $f : A \to S^2_R$ (with $A \subset S^2_R$) is analytic at $p_0 \in A$ iff for at least one (hence all) of the up to 4 possible representative maps from chart to chart is analytic at the corresponding z_0 or w_0

$$\begin{tikzcd}
 & S^2_R \arrow[rrr, "f"] & & & S^2_R & \\
\mathbb{C}_z \arrow[ur, "i_z"] & & \mathbb{C}_w \arrow[ul, "i_w"'] & \mathbb{C}_z \arrow[ur, "i_z"] & & \mathbb{C}_w \arrow[ul, "i_w"']
\end{tikzcd}$$

$i_z^{-1} \circ f \circ i_z$
$i_z^{-1} \circ f \circ i_w$
$i_w^{-1} \circ f \circ i_z$
$i_w^{-1} \circ f \circ i_w$

<u>Examples</u> f is analytic at $z=\infty$ means

$$\begin{cases} f(\tfrac{1}{w}) \text{ is analytic at } w=0, & \text{in case } f(\infty) \text{ is finite in the } z\text{-plane} \\ \dfrac{1}{f(\frac{1}{w})} \text{ is analytic at } w=0, & \text{in case } f(\infty)=\infty. \end{cases}$$

<u>Theorem</u>: Let $f(z)=\dfrac{p(z)}{q(z)}$ be a rational function

Then f extends to be an analytic map $f: S^2_R \to S^2_R$

<u>pf</u>: Only need to check what happens at $z=\infty$ and at zeroes of q. (assume p,q have no common factors)

e.g. If $f(\infty)\neq\infty$, then $\deg p \le \deg q$.

$$f(z)=\frac{a_n z^n + \cdots + a_1 z + a_0}{b_N z^N + \cdots \; b_1 z + b_0} \qquad n \le N$$

$$f(\tfrac{1}{w}) = \frac{a_n(\frac{1}{w})^n + \cdots + a_1(\frac{1}{w}) + a_0}{b_N(\frac{1}{w})^N + \cdots + b_1(\frac{1}{w}) + b_0} = \frac{w^N(a_n\frac{1}{w^n} + \cdots + a_0)}{b_N + \cdots}$$

all other cases are analogous

is analytic at $w=0$

<u>Corollary</u> FLT's $f(z)=\dfrac{az+b}{cz+d}$ are conformal bijections of S^2_R to itself.

<u>Theorem</u> FLT's are the only invertible analytic maps from S^2_R to itself.

<u>pf</u>: Let $g: S^2_R \to S^2_R$ be a conformal diffeomorphism.

Compose g with an FLT f s.t.

$f\circ g(\infty)=\infty$

Then $f\circ g: \mathbb{C}_z \to \mathbb{C}_z$ is a conformal bijection.

Thus $f\circ g = T$, $T(z)=az+b$ ($a\neq 0$) HW

$\Rightarrow g = f^{-1}\circ T$ is an FLT! ∎