Math 4200
Wednesday Nov. 28

- Finish ! part of Riemann mapping theorems, Monday notes

- Then study fractional linear transformations (FLT's), which provide the missing degrees of freedom in Monday's examples, and also have many other important applications.

an FLT $f(z)$, $f: \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is defined by

$$f(z) = \frac{az+b}{cz+d} \qquad \det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{vmatrix} a & b \\ c & d\end{vmatrix} = ad-bc \neq 0, \qquad a,b,c,d \in \mathbb{C}$$

($\det = 0$ $\Rightarrow f$ const!)

Example $f(z) = az+b = \dfrac{az+b}{0z+1}$. You will show in HW these are the only $\overset{\text{injective}}{\text{1-1}}$ conformal maps defined on $\mathbb{C}$!

Exercise 1: Why is there no conformal bijection $f: \mathbb{C} \to D(0;1)$?

Properties of FLT's:

Exercise 2 Show that if

$$f(z) = \frac{az+b}{cz+d}$$

$$g(w) = \frac{\alpha w+\beta}{\gamma w+\delta}$$

Then $g(f(z)) = \dfrac{Az+B}{Cz+D}$,

where $\begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}$

The matrix of the composition is the product of the matrices!

Geometers would say: The group $SL(2,\mathbb{C})$ ("$\underset{\uparrow\,\det=1}{\text{Special}}$ linear group of 2×2 matrices with $\mathbb{C}$ coeffs") acts on $\mathbb{C}\cup\{\infty\}$.

Exercise 3 : Use Exercise 2 to show that

if $f(z) = \dfrac{az+b}{cz+d}$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $f^{-1}(w) = \dfrac{dw-b}{-cw+a}$ $\quad Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$!

Corollary FLT's are bijections of $\mathbb{C} \cup \{\infty\}$ (you can check ∞ separately).

Notice, if $f(z) = \dfrac{az+b}{cz+d}$

$$f'(z) = \frac{a(cz+d)-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0 \qquad (z \neq -\tfrac{d}{c})$$

so LFT's are conformal.

Exercise 4 : FLT's transform circles and lines into circles or lines

pf: true for $T_1(z) = z+a$ (translation)

$T_2(z) = cz$ (scaling and rotation)

$T_3(z) = \frac{1}{z}$ (inversion)

4a) Check claim for T_3: Use the fact that a circle or line is the solution set to an equation of the form $A(x^2+y^2) + Bx + Cy + D = 0$.

4b) Show that compositions of T_1, T_2, T_3 yield the general FLT $f(z) = \dfrac{az+b}{cz+d}$

Hint: do long division on $f(z)$, in case $a \neq 0$.

Thus FLT's do transform circles and lines into circles or lines! ■

Notice that

$$f(z) = \left(\frac{z-a}{z-b}\right)\left(\frac{c-b}{c-a}\right) d$$

maps
$a \mapsto 0$
$b \mapsto \infty$
$c \mapsto d$ (could take $d=1$).

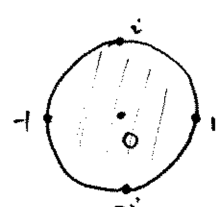
- Composing such f & their inverses lets you construct a LFT mapping any triple of distinct points to any other triple of distinct points.
- Since circles and lines are uniquely determined by any 3 distinct points on them, this lets you construct LFT's mapping any circle or line to any other circle or line

<u>Exercise 5</u> Find an LFT mapping the unit disk to the upper half plane,

by mapping
$-1 \to 0$
$1 \to \infty$
$-i \to 1$

and then adjusting if necessary

(you may wish to appeal to magic thm on next page)



Magic topology theorem for conformal maps

Recall, the Riemann sphere $S^2_R = \mathbb{C} \cup \{\infty\}$.

<u>Theorem</u>: Let $A \subset S^2_R$ open & connected, $\bar{A} \neq S^2_R$.

Let $B \subset S^2_R$ s.t. B and $S^2_R \setminus B$ are both open and connected.

Let $f: A \to S^2_R$ conformal

$f: \bar{A} \to S^2_R$ continuous

$f(\partial A) \subset \partial B$

and let

$\exists\, z_0 \in A$ s.t. $f(z_0) \in B$.

Then $f(A) = B$ & $f(\partial A) = \partial B$, so if f is 1-1 (on A) then A & B are conformally equivalent!

<u>Remark</u>: Roughly speaking, this says con<u>formal</u> maps must fill up the insides, and can't leak over the edges. This is far from true for arbitrary maps!

<u>Proof</u>: Since neither $\bar{A}$ nor $\bar{B}$ is S^2_R, we may pick $z^* \notin \bar{A}$, $w^* \notin \bar{B}$ $(z^*, w^* \in \mathbb{C})$,

then $\tilde{f} = \chi_2 \circ f \circ \chi_1$, $\chi_1(z) = \frac{1}{z - z^*}$, $\chi_2(w) = \frac{1}{w - w^*}$

$\tilde{f}: \underbrace{\chi_1^{-1}(A)}_{\text{bounded in } \mathbb{C}} \to \mathbb{C}$ is cont, $\tilde{f}(\chi_1^{-1}(z_0)) \in \underbrace{\chi_2(B)}_{\text{bounded in } \mathbb{C}}$

i.e., we may assume

A, B, are both bounded domains in $\mathbb{C}$.

<u>Step 1</u>: $B \subset f(A)$: Let $D := f(A) \cap B$, $\neq \emptyset$, since $f(z_0) \in D$.

D open: If $w_1 = f(z_1) \in D$, then f conformal at $z_1 \Rightarrow$ local inverse $f^{-1}: D(w_1; r_1) \to D(z_1, \rho_1) \subset A$ $\exists$. ■

D closed: If $\underset{\subset D}{\{w_k\}} \to w \in B$, pick $\{z_k\} \subset A$ s.t. $f(z_k) = w_k$.

$\bar{A}$ compact $\Rightarrow \exists$ subs $\{z_{k_j}\} \to z \in \bar{A}$

$\Rightarrow \{f(z_{k_j})\} = \{w_{k_j}\} \to w$

so $f(z) = w$. Since $w \notin \partial B$, $z \notin \partial A$.

Thus $w \in D$.

Thus $D = B$ ■

<u>Step 2</u>: $f(A) \subset \bar{B}$:

$\mathbb{C} \setminus \bar{B}$ is connected, and if $E := f(A) \cap (\mathbb{C} \setminus \bar{B})$ then E is open and closed (see step 1).

But $f(\bar{A})$ is compact, hence bounded, in $\mathbb{C}$, so cannot be all of $\mathbb{C} \setminus \bar{B}$. Thus E is empty! ■

<u>Step 3</u>: $f(A) \subset B$:

The argument of step 1 shows $f(A)$ is open.

Thus $f(A) \subset \bar{B} \Rightarrow f(A) \subset B$

Thus $f(A) = B$!

Thus $f(\bar{A}) = \bar{B}$.

Thus $F(\partial A) = \partial B$! ■