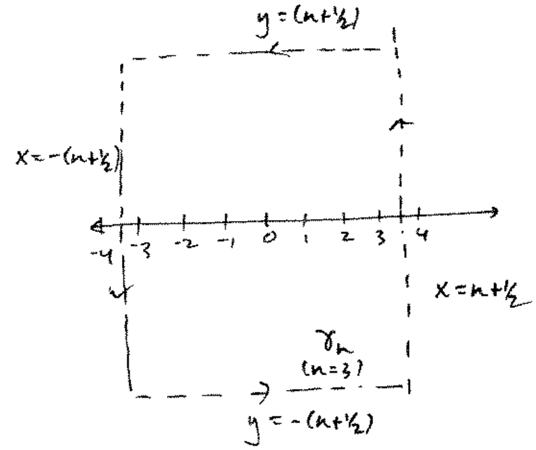
Math 4200
Monday Nov. 26

§4.4 ; begin §5.1



On Friday we proved that if

(i) $f : \mathbb{C} \setminus \{z_1, \ldots z_k\} \to \mathbb{C}$ analytic

(ii) $\displaystyle\int_{\gamma_n} f(z)\,\pi\cot\pi z\,dz \longrightarrow 0$ as $n \to \infty$

Then
$$\lim_{n\to\infty} \sum_{j=-n}^{n} f(j) = -\sum_{\ell=1}^{k} \operatorname{Res}(f(z)\,\pi\cot\pi z\,;\,z_\ell)$$

and used this to derive magic series formulas,
e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ etc.

<u>We claimed</u> $\max\{|\cot\pi z|,\ z\in\gamma_n\} \le 2$ for n large.

Details:
$$\cot\pi z = \frac{\cos\pi z}{\sin\pi z} = \frac{\cos\pi x\,\overbrace{\cos i\pi y}^{\cosh\pi y} - \sin\pi x\,\overbrace{\sin i\pi y}^{i\sinh\pi y}}{\sin\pi x\,\underbrace{\cos i\pi y}_{\cosh\pi y} + \cos\pi x\,\underbrace{\sin i\pi y}_{i\sinh\pi y}}$$

$$|\cot\pi z|^2 = \frac{\cos^2\cosh^2 + \sin^2\overbrace{\sinh^2}^{\cosh^2-1}}{\sin^2\cosh^2 + \cos^2\underbrace{\sinh^2}_{\cosh^2-1}} = \frac{\cosh^2\pi y - \sin^2\pi x}{\cosh^2\pi y - \cos^2\pi x}$$

vertical sides: $x=\pm(n+\frac{1}{2})$; $|\cot\pi z|^2 = \dfrac{\cosh^2\pi y - 1}{\cosh^2\pi y} < 1$

horizontal sides $y=\pm(n+\frac{1}{2})$;
$|\cot\pi z|^2 \le \dfrac{\cosh^2\pi(n+\frac{1}{2})}{\cosh^2\pi(n+\frac{1}{2})-1} < 2 \quad \forall n.$

Also, (improvement of Friday claim)

<u>Theorem</u>: Let $f(z)$ as above.
If $\exists\, M, R$ s.t. $|f(z)| \le \dfrac{M}{|z|}$ $\forall\, |z| \ge R$
Then $\displaystyle\int_{\gamma_n} f(z)\,\pi\cot\pi z\,dz \longrightarrow 0$ as $n\to\infty$.

<u>pf</u>. Consider the Laurent expansion for $f(z)$ in $|z| \ge R$; $\displaystyle f(z) = \frac{b_1}{z} + \sum_{j=2}^{\infty} \frac{b_j}{z^j}$
(no powers z^n, $n \ge 0$ appear, by formula for Laurent coeff's)
$= \dfrac{b_1}{z} + g(z)$, $|g(z)| \le \dfrac{C}{|z|^2}$ for $|z| \ge 2R$.

Then
$$\int_{\gamma_n} f(z)\,\pi\cot\pi z\,dz = \underbrace{\int_{\gamma_n} \overbrace{\frac{b_1}{z}}^{\text{odd}}\ \overbrace{\cot\pi z}^{\text{odd}}\ \overbrace{dz}^{\text{odd}}}_{=0\ \text{(or use residue calc)}} + \underbrace{\int_{\gamma_n} g(z)\,\pi\cot\pi z\,dz}_{|\ | \le \frac{C}{(n+\frac{1}{2})^2}\cdot 2\cdot 4(n+\frac{1}{2}) \longrightarrow 0 \text{ as } n\to\infty}$$ ■

Other interesting series recovers $\pi \cot \pi z$:

Let $f(z) = \frac{1}{z - z_0}$, satisfies previous page thm. $(z_0 \neq n)$

Thus

$$\lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{j - z_0} = - \mathrm{Res}\left(\frac{1}{z - z_0} \pi \cot \pi z ;\ z_0 \right)$$

$$= - \pi \cot \pi z_0$$

$$\therefore \quad \cot \pi z = \lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{z - j}$$

$$= \frac{1}{z} + \lim_{n \to \infty} \left[\sum_{j=1}^{n} \left(\frac{1}{z - j} + \frac{1}{j} \right) + \sum_{j=1}^{n} \frac{1}{z + j} - \frac{1}{j} \right]$$

these series converge absolutely (uniformly absolutely on compact sets)

since, e.g. $\frac{1}{z - j} + \frac{1}{j} = \frac{z}{(z - j) j}$; $| \ | \sim \frac{1}{j^2}$

$$\therefore \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z - n} + \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{z + n} - \frac{1}{n} .$$

(See HW for application).

There is a more general version of this result, see Theorem 4.4.5 page 311

Chapter 5: Conformal maps

This is actually an in depth return to the ideas we began the course with:

Recall the chain rule for curves:

f analytic at z_0
$\gamma: J \to \mathbb{C}$ diffble, $\gamma(t_0) = z_0$

$\Rightarrow (f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$

so $f'(z_0) \neq 0 \Rightarrow$ tangent vectors at z_0 are mapped to tangent vectors at $f(z_0)$, all of which are scaled by $|f'(z_0)|$ and rotated by $\arg(f'(z_0))$.

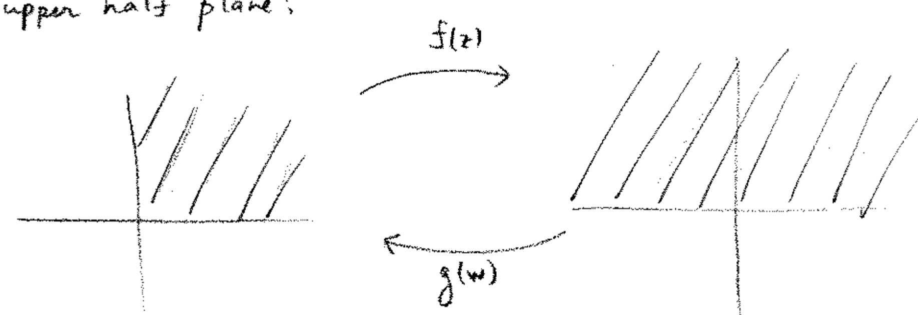
We call(ed) a map conformal iff it has this infinitesimal (i.e. on tangent vectors) rotation and scaling property $\forall z \in A$ (the domain)
By Cauchy-Riemann this is equivalent to
$f: A \to \mathbb{C}$ analytic and $f'(z) \neq 0 \ \forall z \in A$

In chapter 5 we are interested in finding bijective conformal maps between open subsets A, B of $\mathbb{C}$ (Then A and B are called conformally equivalent).
(Applications include PDE's & geometry)

Of course we remember many examples from chapter 1...

example:

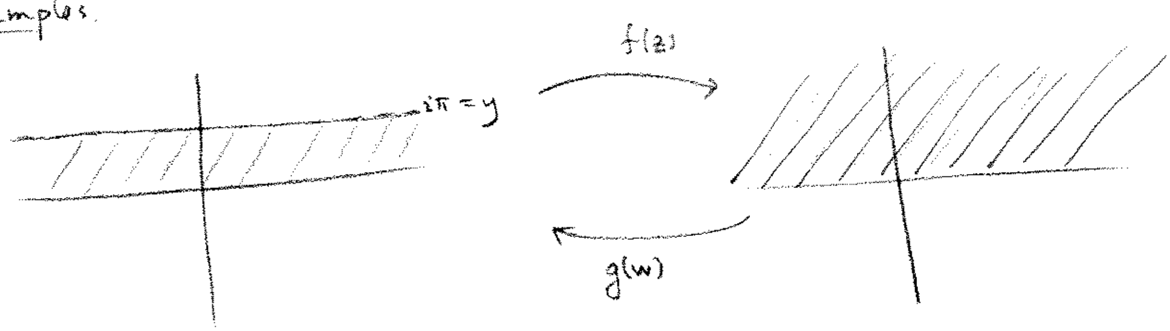
how many conformal bijections can you find between the 1st quadrant and the upper half plane?



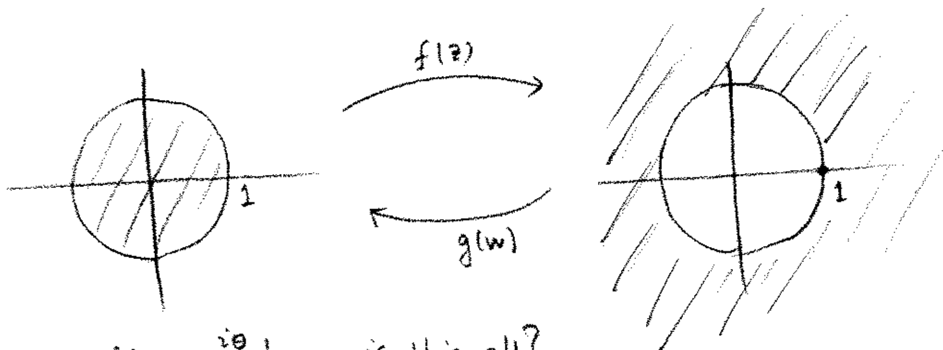
$f(z) = z^2$
$f(z) = az^2 + b \quad a > 0, b$ real

is this all?

examples.



$f(z) = ae^{z} + b$ $\quad a > 0$, $b \in \mathbb{R}$ $\quad$ is this all?



$f(z) = e^{i\theta}\frac{1}{z}$ $\quad$ is this all?

note $f(0) = \infty$, so this map should really be considered on the Riemann sphere.

Riemann Mapping Theorem (version 1)

Let $A \neq \mathbb{C}$ be open and simply connected, $z_0 \in A$

Then $\exists !\, f: A \to D(0,1)$ s.t. f is a conformal bijection, $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}$, $(\arg = 0)$ $f'(z_0) > 0$

3 real degrees of freedom

proof of existence is hard - consult more advanced text, or maybe next week.

proof of uniqueness is "easy":

Suppose $\exists\, f_1, f_2$ as above. consider

$g(z) = f_2 \circ f_1^{-1} : D \to D$.

then $g(0) = 0$, so $G(z) = \begin{cases} \frac{g(z)}{z} & z \neq 0 \\ g'(0) & z = 0 \end{cases}$ is analytic.

$|G(z)| \leq \frac{1}{r}$ on $|z| = r \Rightarrow$ for $|z| \leq r$, by max mod. principle

as $r \to 1 \Rightarrow |G(z)| \leq 1$ in $D(0,1)$

$\Rightarrow |g(z)| \leq |z|$ in $D(0,1)$

[we are reproducing Schwarz Lemma]

But by symmetric argument, $g^{-1}(z) = f_1 \circ f_2^{-1}$

also satisfies $|g^{-1}(z)| \leq |z|$ $\forall z \in D(0,1)$

$\Rightarrow |g^{-1}(g(z))| \leq |g(z)|$

$|z| \leq |g(z)|$

$\Rightarrow |g(z)| = |z|$

$\Rightarrow |G(z)| \equiv 1$

$\Rightarrow G(z) = \text{const} = e^{i\theta}$

but $g'(0) = (f_2 \circ f_1^{-1})'(0)$

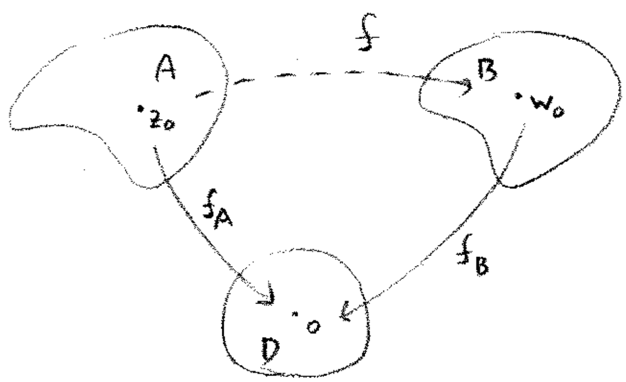
$= f_2'(z_0)\underbrace{(f_1^{-1})'(0)}_{\frac{1}{f_1'(z_0)}}$ is real positive # $\Rightarrow g(z) = z$
$\Rightarrow f_2 = f_1$!

<u>Riemann Mapping Theorem</u> (version 2)

Let $A, B \subset \mathbb{C}$ be simply connected and not all of $\mathbb{C}$.
Let $z_0 \in A$, $w_0 \in B$

Then $\exists!$ $f: A \to B$ conformal bijection s.t. $f(z_0) = w_0$
$\arg f'(z_0) > 0$

<u>pf</u>:



$\exists$ f_A, f_B by RMT$_1$

Let $f := f_B^{-1} \circ f_A$

$f'(z_0) = (f_B^{-1})'(0)\, f_A'(z_0) \in \mathbb{R}^+$

If $\exists$ f_2 with identical props as f
then $f_B \circ f_2 \circ f_A^{-1}$ maps disk to itself
$0 \to 0$
$\text{deriv}(0) \in \mathbb{R}^+$
$\Rightarrow f_B \circ f_2 \circ f_A^{-1}(z) = z$
as on page 4.

■

So did we get all the conformal maps in our 3 examples?

<u>ans</u>: not in any of them!