Math 4200

Wednesday Nov. 21

HW for Friday Nov 30

4.4 1, 2, 3, 4, 5, 8, 9 (9 is impossible, as far as I know.)

5.1 10, 11, 12

We can try one of the §4.3 problems due today.

Also, $\int_0^\infty \frac{\sin x}{x}\,dx$, from pages 7-8 on Wed Nov. 14 notes

My favorite contour integrals lead to magic formulas for some infinite series:

Consider $f(z)\pi\cot(\pi z)$, where $f(z)$ is defined on $\mathbb{C}\setminus\{z_1,\dots z_k\}$ and analytic.

<u>Example</u> $\frac{1}{z^2}\pi\cot\pi z = g(z)$, i.e. $f(z)=\frac{1}{z^2}$

$\pi\cot\pi z$ has simple poles at each integer

so, if $f(n)\neq 0$, $\operatorname{Res}\left(f(z)\pi\cot\pi z\,;n\right) = \left.\frac{f(n)\pi\cos\pi n}{(\sin\pi z)'}\right|_{z=n} = f(n)\,\frac{\pi\cos\pi n}{\pi\cos\pi n} = f(n).$

for $f(z)=\frac{1}{z^2}$, we get $\operatorname{Res}\left(\frac{1}{z^2}\pi\cot\pi z;n\right)=\frac{1}{n^2}$ for $n\neq 0$

For $n=0$ the pole at $z=0$ of $\frac{1}{z^2}\pi\cot\pi z$ is of order 3.
We can use MAPLE to find the residue:

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> series(Pi*cot(Pi*z),z,10);
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$$z^{-1}-\frac{\pi^2}{3}z-\frac{\pi^4}{45}z^3-\frac{2\pi^6}{945}z^5-\frac{\pi^8}{4725}z^7-\frac{2\pi^{10}}{93555}z^9+O(z^{10})$$

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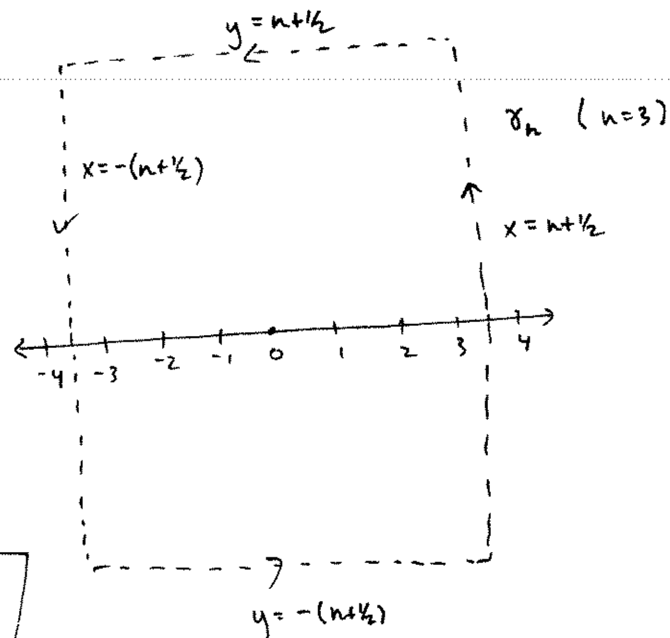
$\Rightarrow \operatorname{Res}\left(\frac{1}{z^2}\pi\cot\pi z;0\right) = -\pi^2/3$



Now, consider $\int_{\gamma_n} f(z)\pi\cot\pi z\,dz$:

γ_n traces a square, thru

$x=\pm(n+\frac{1}{2})$
$y=\pm(n+\frac{1}{2})$

An immediate! consequence of the residue theorem is:

<u>Theorem</u> If $\int_{\gamma_n} f(z)\pi\cot\pi z\,dz \to 0$ as $n\to\infty$,

then $\lim \sum_{\substack{j=-n\\ j \text{ not a singular pt of } f(z)}}^{n} f(j) = -\sum_{z_k \text{ sing pts of } f} \operatorname{Res}\left(f(z)\pi\cot\pi z;z_k\right)$

Check!

So, when can we ensure

$$\int_{\gamma_n} f(z)\, \pi \cot \pi z \, dz \longrightarrow 0 \quad \text{as } n \to \infty.$$

<u>Estimates</u> : $\cot \pi(z+1) = \dfrac{\cos \pi(z+1)}{\sin \pi(z+1)} = \dfrac{-\cos \pi z}{-\sin \pi z} = \cot \pi z$, so $\cot \pi z$ has real period $x = 1$

Thus, along each γ_n's vertical sides $\cot\left(\pi\left(n+\tfrac{1}{2}+iy\right)\right) = \cot\left(\pi\left(\tfrac{1}{2}+iy\right)\right)$

$$= \frac{\cos\left(\frac{1}{2}\pi + \frac{i\pi y}{2}\right)}{\sin\left(\frac{1}{2}\pi + \frac{i\pi y}{2}\right)} = \frac{-\sin\left(\frac{i\pi y}{2}\right)}{\cos\left(\frac{i\pi y}{2}\right)}$$

so $|\cdot| = \dfrac{\left|\sinh \frac{\pi y}{2}\right|}{\cosh \frac{\pi y}{2}} \le 1 \quad \forall y.$

Along horizontal sides.

$$\frac{\cos(\pi(x \pm i(n+\frac{1}{2})))}{\sin(\pi(x \pm i(n+\frac{1}{2})))} = \frac{\cos \pi x \overbrace{\cos\left(\tfrac{i}{\pi}(n+\tfrac{1}{2})\right)}^{\cosh((n+\frac{1}{2})\pi)} \mp \sin \pi x \sin\left(i^{\pi}(n+\tfrac{1}{2})\right)}{\cos \pi x \sin\left(\pm i\pi(n+\tfrac{1}{2})\right) + \sin \pi x \cos\left(i\pi(n+\tfrac{1}{2})\right)}$$

$$|\quad|^2 = \frac{\cos^2 \pi x \cosh^2((n+\frac{1}{2})\pi) + \overbrace{\sin^2 \pi x}^{1-\cos^2 \pi x} \sinh^2 \pi(n+\frac{1}{2})}{\cos^2 \pi x \sinh^2((n+\frac{1}{2})\pi) + \sin^2 \pi x \cosh^2 \pi(n+\frac{1}{2})}.$$

$\lim_{n \to 0} \left|\dfrac{\cosh Cn}{\sinh Cn}\right| = 1$, so $|\cot \pi z| \to 1$ uniformly on top & bottom edges, as $n \to \infty$

<u>Summary</u> : For n large,

$|\cot \pi z| \le 2$ on γ_n.

<u>Corollary</u>: If $|f(z)| \le \dfrac{C}{|z|^p}$, $p > 1$ and $|z| \ge M$

then $\displaystyle\int_{\gamma_n} |f(z)||\pi \cot \pi z| \, |dz| \le \frac{C}{n^p} 2\pi \, 4(2n+1) \longrightarrow 0$ as $n \to \infty$.

<u>Improvement</u> (even/odd).

If $f(z) = k(z) + \tilde{f}(z)$ where $|\tilde{f}(z)| \le \dfrac{C}{|z|^p}$, $p > 1$, $|z| \ge M$

and $k(z)$ is <u>odd</u>, then $\displaystyle\int_{\gamma_n} \underbrace{f(z)}_{(k+\tilde{f})} \overbrace{\pi \cot \pi z}^{\text{odd}} \overbrace{dz}^{\text{odd on } \gamma_n} = \int_{\gamma_n} \tilde{f}(z) \pi \cot \pi z \, dz \longrightarrow 0$ as $n \to \infty$

odd

$f(z) = \frac{1}{z^2}$, page 2 & page 3

$$\Rightarrow \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n^2} = \operatorname{Res}\left(\frac{1}{z^2}\pi\cot\pi z; 0\right) = \frac{\pi^2}{3}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

<u>Exercise</u> Use the page 2 Maple cheat to deduce

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} =$$

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} =$$