Math 4200
Fri. Nov. 16

Exam Monday : 2.4 – 4.2
Review Session tomorrow (Saturday) 2:00 – 3:30
LCB 218, or nearby.

2.4 Cauchy Integral Formula
- also for derivatives
- Liouville's Thm
- Fund. Thm Alg.
- Morera

2.5 Max. Mod. Thm. & harmonic fcns
- Mean value property
- harmonic functions, legal proof that conjugates exist in simply connected domains
  - Dirichlet problem
  - Poisson Integral formula. (I would provide this.)

3.1 Convergent seq's & series of analytic fcns.
- unif. limits of analytic are analytic
- Weierstrass M test

3.2 Power Series & Taylor's Thm
- radius of convergence
- term by term diff
- uniqueness
- analytic $\Leftrightarrow$ power series
- multiplication of series
- examples

3.3 Laurent series
- derivation, uniqueness
- isolated singularities
- residue

4.1 Calculating residues
- multiplication of Laurent
- using Taylor
- using partial fractions
- (Table will be provided.)

4.2 Residue Thm
- statement & proof.
- examples.
- residues at ∞.

HW questions?
Wed notes example good to finish.
Also, proof that it's O.K. to multiply two infinite
Laurent series term by term.

Math 4200

November 16, 200~~6~~7

PRACTICE EXAM 2

Each complete problem below is worth 30 points. 10 pts for your name! Choose any three out of the six problems to do. If you try more than three problems, indicate clearly which three you want graded. This exam is closed book and closed note, except for the residue ~~and contour integral~~ tables which I've provided. Show complete work for complete credit. Justify all steps in theorem proofs. Good Luck!

1a) Prove the Cauchy Integral Formula, which states that if γ is a piecewise C^1 closed curve homotopic to a point in an open region A, and if $f(\zeta)$ is analytic for all ζ in A, then

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i\, f(z)\, \mathrm{Index}(\gamma; z)$$

(20 points)

1b) What is the corresponding formula for the nth derivative of f at z? Explain very briefly how this formula is derived.

(5 points)

1c) Use the Cauchy formula for the first derivative of f, using circle contours with radius R about arbitrary points z, and then letting R approach infinity, to prove that every bounded entire function is constant (Liouville's Theorem).

(5 points)

2) We proved that if f(z) is analytic in an annulus $r < |z - z_0| < R$, then it has a Laurent series

$$f(z) = \left(\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}\right) + \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)$$

And, conversely, we proved that any series of this type has a natural annulus of convergence so that each half of the Laurant series converges uniformly absolutely on compact subannuli

$$r < \rho \le |z - z_0| \le P < R.$$

2a) Let γ be a circle of radius greater than r and less than R, centered at z_0 and oriented counterclockwise. Prove that

$$\int_\gamma f(z)\, dz = 2\pi i\, b_1$$

(15 points)

(You may quote theorems which allow you to interchange a limit process with integration.)

2b) Extend your argument from 2a to show that every Laurent coefficient b_n, a_n is uniquely determined by the analytic function f(z), by finding a formula for each such coefficient.

(15 points)

3a) Find the first four non-zero coefficients in the Laurent series for

$$f(z) = \frac{e^z}{z \sin(z)}$$

at $z_0=0$.

(20 points)

3b) What is the outer radius of the largest punctured disk about the origin in which the Laurant series for f(z) converges? Explain!

(5 points)

3c) Let γ be a circle centered at the origin, oriented counterclockwise. Compute

$$\int_\gamma f(z)\, dz$$

(5 points)

4) Evaluate the following integrals

a) $\displaystyle\int_{|z|=1/2} \frac{1}{(1-z)^3}\,dz$ (10 pts)

b) $\displaystyle\int_{|z-1|=1/2} \frac{1}{(1-z)^3}\,dz$ (10 pts)

c) $\displaystyle\int_{|z-1|=6} \frac{e^z}{(1-z)^3}\,dz$ (10 pts)

5) a) Derive the Taylor series for $(1-w)^p$ at $w_0=0$. (15)

b) What is the radius of convergence of the series above (when p is not a positive integer) (5)

c) Find $\displaystyle\int_{|z|=2} \frac{1}{\sqrt{z^2-1}}\,dz$ where argument is standard branch. (10)

Hint: $\dfrac{1}{\sqrt{z^2-1}} = \dfrac{1}{z}\left(1-\dfrac{1}{z^2}\right)^{-1/2}$

6) a) State and prove the maximum modulus principle, for an analytic function f defined on an open, bounded, connected set A, where f extends continuously to the closure of A (20 points)

b) Find the maximum value of $|f(z)| = |\cos z|$ on the square $\{z = x+iy \mid |x|\le 1, |y|\le 1\}$ (10 points)