Math 4200
Wed 14 Nov.

Residue Theorem

Let $f : A \to \mathbb{C}$ analytic except at $\{z_1, \dots z_N\} \subset A$

$\gamma : I \to A$ closed curve (p.w. C') homotopic to a point in A

for the record, A need not be simply connected. You just need γ homotopic to a point to apply deformation thm. (My bad on Friday)

Then $$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)\, I(\gamma; z_j)$$

proof: Expand f in Laur. series at each z_j:

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_j)^n}}_{S_j(z)} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_j)^n}_{A_j(z)} \qquad \text{in } D(z_j, r_j) \setminus \{z_j\}$$

$S_j(z)$ converges to an analytic fcn in all of $\mathbb{C} \setminus \{z_j\}$

Let $g(z) := f(z) - \sum_{j=1}^{N} S_j(z)$. Then the singularities of g at z_j are removable, since $f(z) - S_j(z) = A_j(z)$ is analytic near z_j (and so is $-\sum_{\ell \neq j} S_\ell(z)$).

So g (extends to be) analytic in A

Since γ homotopic to a point in A,

$$\int_\gamma g(z)\,dz = 0 = \int_\gamma f(z) - \sum_{j=1}^{N} \underbrace{\int_\gamma S_j(z)\,dz}_{\int \sum_{n=1}^{\infty} \frac{b_n}{(z-z_j)^n} dz}$$

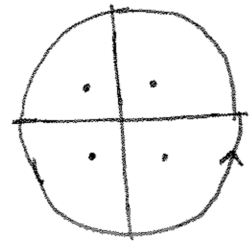
$$\underbrace{\sum_{n=1}^{\infty} \int_\gamma \frac{b_n}{(z-z_j)^n} dz}_{= b_{+1} \int_\gamma \frac{1}{z-z_j} dz} = b_{+1}(2\pi i\, I(\gamma; z_j))$$

$$0 = \int_\gamma f(z)\,dz - 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f; z_j)\, I(\gamma; z_j)$$

∎

Example @ Prof. Treibergs:

$$\oint_{|z|=3} \frac{z}{z^4+1}\,dz \qquad \text{poles } z^4+1=0,\quad z^4=-1,\quad z = e^{i\pi/4},\ e^{3i\pi/4},\ e^{-i\pi/4},\ e^{-3i\pi/4}$$



simple poles,

$$f = \frac{g}{h},\quad Res(f, z_0) = \frac{g(z_0)}{h'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2}$$

$$Res(f; e^{i\pi/4}) = \frac{1}{4e^{i\pi/2}} = -\frac{1}{4}i$$

$$Res(f; e^{3i\pi/4}) = \frac{1}{4e^{3i\pi/2}} = \frac{1}{4}i \qquad 2\pi i\left(\sum_{j=1}^{4} res(f; z_j)\right) = \boxed{0}$$

$$Res(f; e^{-3i\pi/4}) = \frac{1}{4e^{-3i\pi/2}} = -\frac{1}{4}i$$

$$Res(f; e^{-i\pi/4}) = \frac{1}{4e^{-i\pi/2}} = \frac{1}{4}i$$

Method 2: Substitution in contour integrals

$z = G(\zeta)$, G analytic, invertible on γ.

Then $\displaystyle\int_\gamma f(z)dz = \int_{G^{-1}(\gamma)} f(G(\zeta))G'(\zeta)d\zeta$ i.e. subs $z = G(\zeta)$, $dz = G'(\zeta)d\zeta$

pf: parameterize: $\displaystyle\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$, $\gamma: [a,b] \to \mathbb{C}$

$$\int_{G^{-1}(\gamma)} f(G(\zeta))G'(\zeta)d\zeta = \int_a^b \underbrace{f(G(G^{-1}(\gamma(t))))}_{f(\gamma(t))}\ \underbrace{G'(\overset{G^{-1}(\gamma(t))}{\downarrow}\cancel{\ \ })\ \overbrace{\underbrace{G^{-1}(\gamma(t))'}_{(G^{-1})'(\gamma(t))\gamma'(t)}dt}^{d\zeta}}_{\text{this product is 1 since it's } G(G^{-1}(z))'}$$

In case at hand, let

$$\zeta = \frac{1}{z} \qquad z = \frac{1}{\zeta}$$

$$\oint_{|z|=3} \frac{z}{z^4+1}\,dz = \oint_{|\zeta|=1/3} \frac{1/\zeta}{(1/\zeta)^4+1}\left(-\frac{1}{\zeta^2}\right)d\zeta = \oint_{|\zeta|=\frac{1}{3}} -\frac{\zeta}{1+\zeta^4}\,d\zeta = \boxed{0};$$ no singularities inside $|\zeta| = 1/3$.

notice orientation change:
if $z = 3e^{it}$
$\zeta = \frac{1}{z} = \frac{1}{3}e^{-it}$

this method generalizes to...

Method 3: Could've used deformation thm with $\gamma_R = \{z \mid |z| = R\}$, let $R \to \infty$, as on 1st midterm

<u>Def</u> If f is analytic in a nbhd of ∞ (i.e. outside $D(0,R)$ some $R>0$),

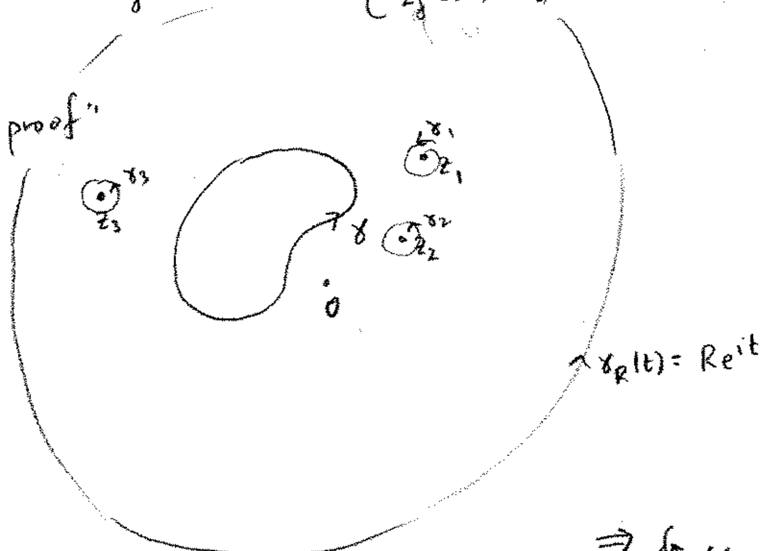
then $\operatorname{Res}(f;\infty) := \operatorname{Res}\left(-\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right); 0\right)$

<u>Thm</u>: Let γ be a simple closed curve in $\mathbb{C}$, traversed once counterclockwise.
Let f analytic along γ with only finitely many singularities <u>outside</u> γ

Then

$$\int_\gamma f(z)\,dz = -2\pi i \left\{ \sum_{z_j \text{ outside } \gamma} \operatorname{Res}(f; z_j) + \operatorname{Res}(f;\infty) \right\}$$

"proof"



Let all $\{z_j\} \subset D(0,R)$

Green's Theorem version of Cauchy

$$\Rightarrow \oint_{\gamma_R} f(z)\,dz - \oint_\gamma f(z)\,dz - \sum_j \oint_{\gamma_j} f(z)\,dz = 0$$

$$\Rightarrow \oint_\gamma f(z)\,dz = \underbrace{\oint_{\gamma_R} f(z)\,dz}_{\zeta = \frac{1}{z}} - 2\pi i \sum_j \operatorname{Res}(f, z_j)$$

$z = Re^{it}$
$\zeta = \frac{1}{R} e^{-it}$

$$\oint_{|\zeta| = \frac{1}{R}} f\left(\frac{1}{\zeta}\right)\left(-\frac{1}{\zeta^2}\right) d\zeta$$

$= -2\pi i \operatorname{Res}(f, \infty)$
check those $-$ signs!

See HW:
§ 4.2 #13,
possibly #15.

Notes on §4.2 #15.

$f(z) = \sqrt{z^2-1} = \sqrt{z-1}\sqrt{z+1}$.

Earlier, we found a natural simply connected domain of analyticity for f:

but would've also done

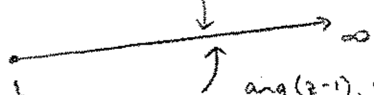
$$f(z) = e^{\frac{1}{2}\left[\log(z-1) + \log(z+1)\right]}$$

$0 < \arg(z-1) < 2\pi$
$0 < \arg(z+1) < 2\pi$

focus here

$\downarrow$

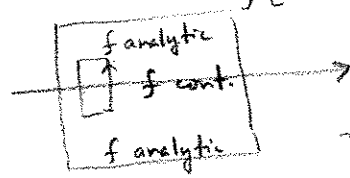
$\arg(z-1), \arg(z+1) \to 0$



$\arg(z-1), \arg(z+1) \to 2\pi$

$\frac{1}{2}[\arg(z-1)+\arg(z+1)] \to 2\pi$.

so f extends continuously from top and bottom along $\{z = x > 1\}$



$\Rightarrow$ f analytic also along real axis

pf: Morera, i.e.
prove rectangle lemma
$\Rightarrow \exists F$ antiderive
$\Rightarrow F' = f$ analytic.

Example 2.4.17
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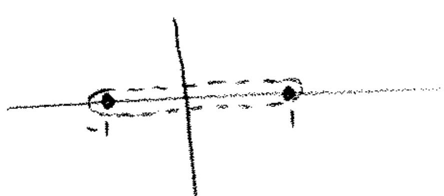
so in fact

$$f(z) = e^{\frac{1}{2}\left[\log(z-1) + \log(z+1)\right]}$$

$0 \le \arg(z-1) \le 2\pi$
$0 \le \arg(z+1) \le 2\pi$
gives analytic fun in



#15. Compute $\displaystyle\oint_{|z|=2} \sqrt{z^2-1}\,dz$ for this f. Two ways possible

1. invert, $z = \frac{1}{3}$, i.e. residue at ∞
2. Laurent series for $\sqrt{z^2-1} = z\left(1-\frac{1}{z^2}\right)^{\frac{1}{2}}$ valid for $|z|>1$

$|w|<1 \Rightarrow (1+w)^p = 1 + pw + \frac{p(p-1)}{2}w^2 + \dots$ $\quad p = \frac{1}{2}$

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Example

$$* \quad \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \quad , \quad \text{e.g. } f \text{ a rational function in } \cos\theta, \sin\theta$$

trick: convert (backwards) into a contour integral on $|z|=1$,
then apply residue thm

$$z = e^{i\theta}$$
$$dz = ie^{i\theta} d\theta \; ; \quad d\theta = \frac{dz}{iz}$$
$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right) \quad \text{on unit circle}$$
$$\sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \tfrac{1}{2}\left(z - \tfrac{1}{z}\right) \quad ''$$

So

$$* = \int_{|z|=1} f\left(\tfrac{1}{2}(z+\tfrac{1}{z}), \tfrac{1}{2}(z-\tfrac{1}{z})\right) \frac{dz}{iz}$$

Exercise 1: Show $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{|a^2-1|}$ for any $a \in \mathbb{R}$, $a \neq \pm 1$

(since denom $= (1-a\cos\theta)^2 + a^2\sin^2\theta$)

integral equals

$$\int_{|z|=1} \frac{1}{1+a^2-\frac{2a}{2}(z+\frac{1}{z})} \frac{dz}{iz} = \int_{|z|=1} \frac{-i\,dz}{-az^2 + z(1+a^2) - a} = \frac{i}{a}\int_{|z|=1} \frac{dz}{z^2 - z\left(\frac{1}{a}+a\right) + 1}$$

$$= \frac{i}{a}\int_{|z|=1} \frac{dz}{(z-a)(z-\frac{1}{a})}$$

$$|a|<1 \Rightarrow \operatorname{Res}\left(\frac{1}{(z-a)(z-\frac{1}{a})}; a\right) = \frac{1}{a-\frac{1}{a}} = \frac{a}{a^2-1}$$

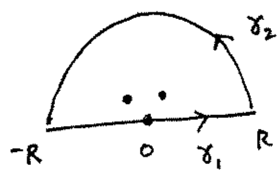
$$\Rightarrow \text{Int} = \frac{i}{a} 2\pi i \frac{a}{a^2-1} = -\frac{2\pi}{a^2-1} = \frac{2\pi}{|a^2-1|}$$

$$|a|>1 \Rightarrow \operatorname{Res}\left(\frac{1}{(z-a)(z-\frac{1}{a})}; \frac{1}{a}\right) = \frac{1}{\frac{1}{a}-a} = \frac{a}{1-a^2}$$

$$\Rightarrow \text{Int} = \frac{i}{\cancel{a}} 2\pi i \left(\frac{\cancel{a}}{1-a^2}\right) = \frac{2\pi}{|a^2-1|} \quad \text{too} \quad \blacksquare$$

Improper integrals

Exercise 2 $\displaystyle\int_0^\infty \frac{dx}{1+x^4} := \lim_{R\to\infty}\int_0^R \frac{dx}{1+x^4} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{1+x^4} := \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$



$\gamma = \gamma_1 + \gamma_2$

$$\int_\gamma \frac{dz}{1+z^4} = \int_{-R}^{R} \frac{dx}{1+x^4} + \int_{\substack{|z|=R \\ \operatorname{Re} z \geq 0}} \frac{dz}{1+z^4} = I_1(R) + I_2(R).$$

$$= 2\pi i \left(\operatorname{Res}(f; e^{i\pi/4}) + \operatorname{Res}(f; e^{3i\pi/4}) \right)$$

since $z^4+1=0$ at $e^{\pm i\pi/4}$, $e^{\pm 3i\pi/4}$, as on page 2

simple poles $f = \frac{g}{h}$, $\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)} = \frac{1}{4z_0^3}$ in our case.

$$\operatorname{Res}(f; e^{i\pi/4}) = \frac{1}{4e^{3i\pi/4}} = \frac{1}{4}e^{-3i\pi/4}$$

$$\operatorname{Res}(f; e^{3i\pi/4}) = \frac{1}{4e^{9i\pi/4}} = \frac{1}{4}e^{-i\pi/4}$$

contour integral ans $= \frac{2\pi i}{4}\left(\cancel{\cos\frac{3\pi}{4}} - i\sin\frac{3\pi}{4} + \cancel{\cos\frac{\pi}{4}} - i\sin\frac{\pi}{4} \right)$

$= +\frac{\pi}{2}\left(2\sqrt{2}\right) = \boxed{\sqrt{2}\,\pi}$

Now, let $R\to\infty$. $\displaystyle I_1(R) \longrightarrow 2\int_0^\infty \frac{dx}{1+x^4}$

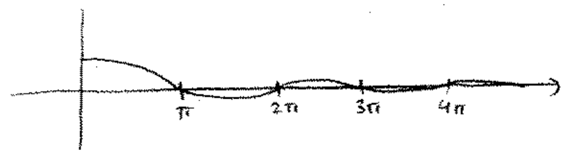
$$|I_2(R)| \leq \int_{\substack{|z|=R \\ \operatorname{Re} z \geq 0}} \frac{1}{R^4-1}\,|dz| = \frac{\pi R}{R^4-1} \to 0 \text{ as } R\to\infty$$

thus $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \boxed{\frac{\pi}{\sqrt{2}}}$

There are lots of interesting improper integrals in §4.3 text & HW.
Here's one of my favorites:

- example

(4) $$\int_0^\infty \frac{\sin x}{x}\,dx$$



$$= \sum_{n=0}^{\infty} \underbrace{\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx}$$

alternating decreasing term series, so convergence at ∞ is no problem, i.e.

$\lim_{N\to\infty} \int_0^N \frac{\sin x}{x}\,dx \;\exists$, even though integral does not converge absolutely.

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \qquad \text{since } \frac{\sin x}{x} \text{ is even.}$$

In this example, for the contour we use, we must consider e^{iz} rather than $\sin z$.

Notice that $\frac{e^{iz}}{z}$ has trouble near $z=0$ however (in the real part).

in this example we could just focus on Im part, but, also a good opportunity to introduce "principal value"

If f has an isolated (real) singularity at $x_0 \in \mathbb{R}$, $x_0 \in [a,b]$

then $$\text{P.V.} \int_a^b f(x)\,dx := \lim_{\varepsilon\to 0} \int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx$$

notice the symmetric way in which x_0 is avoided.

example $\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx$ doesn't strictly make sense, since e.g.

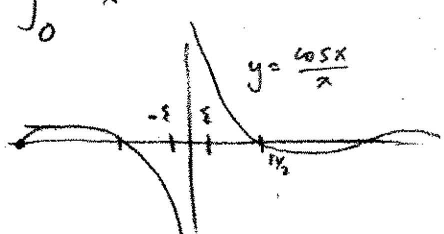
$$\int_0^\varepsilon \frac{\cos x}{x}\,dx = +\infty$$

but $$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx$$

$$= \lim_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} \frac{\cos x}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\cos x}{x}\,dx$$

exactly cancel! since $\frac{\cos x}{x}$ is odd

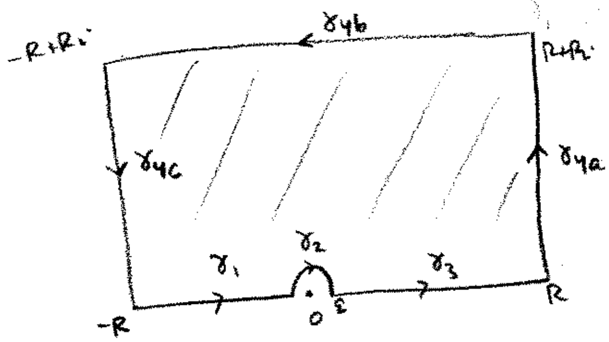
$$= 0.$$



Thus $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\mathrm{Im.}\ \mathrm{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$

$$= \frac{1}{2}\mathrm{Im} \lim_{\varepsilon\to 0}\left(\lim_{R\to\infty}\left(\int_{-R}^{\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx\right)\right)$$

$\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz$



$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)dz = 0.$$

$$\lim_{R\to\infty} \int_{\gamma_4} f(z)dz = 0.$$

γ_{4a}: $\left|\frac{e^{iz}}{z}\right| = \frac{e^{-y}}{|z|} \le \frac{e^{-y}}{R}.$

$\int_{\gamma_{4a}} |f(z)||dz| \le \frac{1}{R}\int_0^R e^{-y}dy \le \frac{1}{R} \to 0$

$\int_{\gamma_{4c}} |f(z)||dz| \le \frac{1}{R} \to 0.$

$\int_{\gamma_{4b}} |f(z)||dz| \le e^{-R}\cdot 2R. \to 0$

✓.

$$\int_{\gamma_2} \frac{e^{iz}}{z}\,dz = \int_{\gamma_2} \frac{1}{z} + \underbrace{\text{analytic near } 0}_{|\ |\le M}\ dz \ \ldots$$

$$= \int_{\gamma_2} \frac{1}{z}dz \pm \underbrace{M\cdot\pi\varepsilon}_{\to 0 \text{ as } \varepsilon\to 0}$$

$$= -\pi i \pm M\pi\varepsilon$$

So $\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = -\int_{\gamma_2} f(z)dz - \int_{\gamma_4} f(z)dz$

$\lim_{R\to\infty}$: $\int_{-\infty}^{\varepsilon} f(z)dz + \int_{\varepsilon}^{\infty} f(z)dz = \pi i \pm M\pi\varepsilon$

$\lim_{\varepsilon\to 0}$: $\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i$

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\pi}$$