1.a) $f : A \to \mathbb{C}$ analytic (A open, connected; as always).

γ p.w. C^1 closed curve, range$(\gamma) \subset A$. γ homotopic to a pt in A, as closed curves.

$z \in A$, $z \notin$ range(γ)

Then

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i\, f(z)\, I(\gamma; z)$$

proof: Consider $g(\zeta) = \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$

g is analytic in $A \setminus \{z\}$ and continuous at z, so by a modification to the rectangle lemma has a local antiderivative, and so the deformation theorem holds for g.

Thus

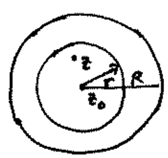
$$\int_\gamma g(\zeta)\, d\zeta = \int_{pt} g(\zeta)\, d\zeta = 0$$

$$//$$

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \underbrace{\int_\gamma \frac{1}{(\zeta - z)}\, d\zeta}_{2\pi i\, I(\gamma; z)} \quad \text{, i.e.} \quad \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z)\, 2\pi i\, I(\gamma; z)$$

∎

1b) Let $0 < r < R$, $|z - z_0| = \rho < r$.



C.I.F. $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} f(\zeta) \frac{1}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} \left(\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \right) d\zeta$$

Note, $w = \frac{z - z_0}{\zeta - z_0}$ has $|w| = \frac{\rho}{r} < 1$

So

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

conv. unif. abs. $\forall \zeta$ s.t. $|\zeta - z_0| = r$

and, also $\displaystyle\sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$ conv. unif. abs. on $|\zeta - z_0| = r$, because $|f(\zeta)| \le M$ there.

$$\left(\sum_{n=0}^{\infty} | \quad | \le \sum_{n=0}^{\infty} \frac{M}{r} \left(\frac{\rho}{r}\right)^n = \frac{M/r}{1 - \rho/r} < \infty \right)$$

thus

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \sum_{n=0}^{\infty} \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

since can interchange Σ and $\int$ when conv. is unif.

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n := \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

∎

2a) A bounded entire fcn is constant.

Pf Let f entire, $|f(z)| \le M < \infty \ \forall z \in \mathbb{C}$.

Let $z_0 \in \mathbb{C}$. $f'(z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta$ C.I.F. for derivs.

so $|f'(z_0)| \le \frac{1}{2\pi} \int_{|\zeta - z_0| = R} \frac{M}{R^2} |d\zeta| = \frac{1}{2\pi} \frac{M}{R^2} 2\pi R = \frac{M}{R}$.

Let $R \to \infty$; $\frac{M}{R} \to 0$.

Hence $|f'(z_0)| = 0 \ \forall z_0 \in \mathbb{C}$

Hence f is constant.

2b) FTA:

By induction, suffices to show $p_n(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

has a root z_0, since then $p_n(z) = (z - z_0) p_{n-1}(z)$.

If $p_n(z)$ has no root, then

$f(z) = \frac{1}{p_n(z)}$ is entire.

Now, $\lim_{|z| \to \infty} \frac{1}{p_n(z)} = \lim_{|z| \to \infty} \frac{1}{z^n} \left(\frac{1}{1 + a_{n-1}/z + \dots + a_0/z^n} \right) = 0 \cdot 1 = 0$.

So $\exists R$ s.t. $|z| > R \Rightarrow \left| \frac{1}{p_n(z)} \right| \le 1$.

Let $M = \max \left\{ \frac{1}{|p_n(z)|}, |z| \le R \right\}$ which exists because $|f(z)|$ is cont. on compact set.

Then $|f(z)| \le \max \{M, 1\} \ \forall z$,
so is bd.

Thus $f(z) \equiv C$ is const. (by Liouville)

$\Rightarrow \frac{1}{p_n(z)} \equiv C \quad (C \ne 0)$

$\Rightarrow p_n(z) \equiv \frac{1}{C}$ is const. But $p_n^{(n)}(z) = n! \ne 0$ $\Rightarrow\!\Leftarrow$.

Thus $p_n(z)$ had a root! ∎

3a) $g(z) = \frac{\sqrt{1+z}}{\sin z}$

$\sqrt{1+z} = (1+z)^{1/2} = 1 + \frac{1}{2} z + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!} z^2 + \dots$

$\sin z = z - \frac{z^3}{6} + \frac{z^5}{5!} - \dots$

$g(z) = \frac{1 + \frac{1}{2} z - \frac{1}{8} z^2 + \dots}{z \left(1 - \frac{z^2}{6} + \frac{z^4}{5!} - \dots\right)}$

so $g(z)$ has a simple pole at $z_0 = 0$,

$\frac{1 + \frac{1}{2} z - \frac{1}{8} z^2 + \dots}{z \left(1 - \frac{z^2}{6} + \frac{z^4}{5!} - \dots\right)} = \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \dots$

$1 + \frac{1}{2} z - \frac{1}{8} z^2 + \dots = (\overset{1}{b_1} + \overset{1/2}{a_0} z + a_1 z^2 + \dots)\left(1 - \frac{z^2}{6} + \frac{z^4}{5!} - \dots\right)$

$1: \quad 1 = b_1$

$z: \quad \frac{1}{2} = a_0$

$z^2: \quad -\frac{1}{8} = -\frac{1}{6} + a_1; \quad a_1 = \frac{1}{6} - \frac{1}{8} = \frac{4-3}{24} = \frac{1}{24}$

so $\boxed{g(z) = \frac{1}{z} + \frac{1}{2} + \frac{1}{24} z + \dots}$

3b) $\sqrt{1+z}$ has radius of conv for its Taylor series $R = 1$

$\frac{1}{\sin z}$ " " Laurent $R = \pi$ $(0 < |z| < \pi)$

therefore Laurent for $\frac{\sqrt{1+z}}{\sin z}$ has annulus of conv. $0 < |z| < 1$

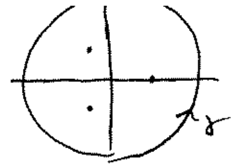
i.e. outer $\boxed{R = 1.}$

Otherwise, $\sqrt{1+z} = (\sin z) g(z)$ would be analytic in a disk of radius > 1, which it's not!

3c) $\int_{|z| = 1/2} g(z) dz = \int_{|z| = 1/2} \frac{1}{z} + \frac{1}{2} + \frac{1}{24} z + \dots \overset{\text{by uniform conv}}{=} \int_{|z| = 1/2} \frac{1}{z} + \underbrace{\int \frac{1}{2} + \int \frac{1}{24} z + \dots}_{\text{all} = 0 \text{ by FTC}}$

$\boxed{= 2\pi i}$

4a) $z^3-1 = (z-e^{2\pi i/3})(z-e^{-2\pi i/3})(z-1)$



so

$f(z) = \frac{2z^2}{z^3-1} = \frac{g(z)}{h(z)}$

has simple poles at $z = e^{2\pi i/3}, e^{-2\pi i/3}, 1$

$\text{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$ if $g(z_0) \neq 0$, $h'(z_0) \neq 0$

$= \frac{2z_0^2}{3z_0^2} = \frac{2}{3}$ for each pole!

$\Longrightarrow \int_\gamma \frac{2z^2}{z^3-1} = 3\left(\frac{2}{3}\right)\cdot 1 \cdot 2\pi i = \boxed{4\pi i}$ $\quad \left(= 2\pi i \sum_{z_j} \text{Res}(f; z_j)\, I(\gamma; z_j) \right)$

4b) $\int_\gamma \frac{2z^2}{z^3-1}\,dz = -2\pi i \left(\sum_{z_j \text{ outside } \gamma} \text{Res}(f; z_j) + \text{Res}(f; \infty) \right)$

$= -2\pi i\, \text{Res}(f; \infty)$

$= -2\pi i\,(-1)\, \text{Res}\left(\underbrace{\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right)}; \zeta = 0 \right)$

$\frac{1}{\zeta^2} \frac{\frac{2}{\zeta^2}}{\left(\frac{1}{\zeta}\right)^3 - 1} = \frac{2/\zeta^2}{\frac{1}{\zeta} - \zeta^2} = \frac{2/\zeta}{1-\zeta^3}$

$= \frac{2}{\zeta}\left(\frac{1}{1-\zeta^3}\right)$

analytic @ $\zeta = 0$

value @ $0 = 1$

so $\text{Res} = 2$

$= -2\pi i(-1)(2)$

$= \boxed{4\pi i}$

(or, $z = \frac{1}{\zeta}$

$dz = -\frac{1}{\zeta^2}d\zeta$

$\oint_{|z|=2} \frac{2z^2}{z^3-1}\,dz = \oint_{|\zeta|=1/2} \frac{2/\zeta^2}{1/\zeta^3 - 1}\left(-\frac{1}{\zeta^2}\right)d\zeta = \oint_{|\zeta|=1/2} -\frac{2}{\zeta}\left(\frac{1}{1-\zeta^3}\right)d\zeta = 2\pi i\,(-2)\underset{I(\tilde{\gamma}, 0) = -1}{(-1)}$

$= \boxed{4\pi i}$

5. a) $\int_{|z|=1} \frac{3}{(z-2)^4}\,dz = -\frac{1}{(z-1)^3}\Big]_1^1 = 0$ F.T.C.

b) $\int_{|z|=3} \frac{3}{(z-2)^4}\,dz = -\frac{1}{(z-1)^3}\Big]_3^3 = 0$ F.T.C.

c) $\int_{|z|=3} \frac{3\sin z}{(z-2)^4}\,dz = 2\pi i\,\text{Res}(f; 2).$

Taylor for $\sin z$ @ $z = 2$

$g(z) = \sin z$
$g' = \cos z$
$g'' = -\sin z$
$g''' = -\cos z$
$\vdots$

$\sin z = \sin 2 + \cos 2\,(z-2) - \frac{\sin 2}{2!}(z-2)^2$
$\boxed{-\frac{\cos 2}{3!}}(z-2)^3 + \cdots$

so $\frac{3\sin z}{(z-2)^4} = \frac{3}{(z-2)^4}$ (expansion above)

so $\text{Res}\left(\frac{3\sin z}{(z-2)^4}; 2\right) = 3\left(\frac{-\cos 2}{3!}\right)$

$= -\frac{1}{2}\cos 2$

$= 2\pi i\left(-\frac{\cos 2}{2}\right)$

$\boxed{= (-\pi\cos 2)\,i}$

6 a) Ratio test for abs conv:

$\frac{|a_{n+1}|}{|a_n|} : \frac{\frac{|n+2||z^{n+1}|}{(n+1)!}}{\frac{|n+1||z^n|}{n!}} = \frac{n+2}{n+1}|z|\frac{1}{n+1} \to 0 \quad \forall z$, as $n \to \infty$,

($\frac{n+2}{n+1} \to 1$)

so $\boxed{R = \infty}$

b) $f(z) = \sum_{n=1}^{\infty} \frac{(n+1)z^n}{n!}$

$\int f(z)\,dz = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n!} = z\sum_{n=1}^{\infty}\frac{z^n}{n!} = z(e^z - 1) + C$ (because series is missing $n=0$ term for e^z)

$\Rightarrow f(z) = D_z\left(z(e^z - 1)\right) = e^z - 1 + ze^z;$

$\boxed{f(z) = (1+z)e^z - 1}$

$\boxed{\text{c) } f(1) = 2e - 1}$

more clever:
$f(z) = \underbrace{\sum_{n=1}^{\infty}\frac{n}{n!}z^n}_{\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}} + \underbrace{\sum_{n=1}^{\infty}\frac{z^n}{n!}}_{e^z - 1}$

$z\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!} = ze^z$