EXAM 2

Math 4200-1   November 19, 2007

Each complete problem below is worth 30 points.   Choose any three out of the six problems to do. You receive 10 points for your good name. If you try more than three problems, indicate clearly which three you want graded. This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

1a) State the Cauchy Integral Formula, which relates the value f(z) of an analytic function on a domain to contour integrals of closed curves which wind around z. Be sure your statement includes the proper hypotheses and conclusions. Then prove the Cauchy Integral Formula (from the Cauchy Theorem, not as a consequence of the Residue Theorem).

(20 points)

1b) Use the Cauchy Integral Formula and clever use of geometric series to prove that if f(z) is analytic in a disk $D(z_0;R)$, then f has a convergent power series expansion in that disk, i.e. derive the series representation.

(10 points)

2a) State and prove Liouville's Theorem, for bounded entire functions. Hint: use the Cauchy Integral Formula for derivatives.

(15 points)

2b) Use Liouville's Theorem to prove the Fundamental Theorem of Algebra, which assserts that every degree n polynomial with complex coefficients factors completely over the complex number field, into n linear factors.

(15 points)

3a) Find the first three non-zero terms in the Laurent series for

$$g(z) = \frac{\sqrt{1+z}}{\sin(z)}$$

centered at $z_0 = 0$. (Use the Taylor series for the numerator and denominator to get started.)

(20 points)

3b) Use general principles to deduce the radius of the largest punctured disk centered at $z_0 = 0$, on which the Laurent series above converges.

(5 points)

3c) Let γ be the circle of radius 1/2 centered at the origin. What is the value of

$$\int_\gamma \frac{\sqrt{1+z}}{\sin(z)}\, dz \,?$$

(5 points)

4a) Let γ be the circle of radius 2, centered at the origin. Use the Residue Theorem to compute

$$\int_{\gamma} \frac{2\, z^2}{z^3 - 1}\, dz$$

(20 points)

(Hint: One of your formulas for residues at simple poles works especially well here!)

4b) Using residue at infinity (which amounts to making the substitution $\zeta = \dfrac{1}{z}$ in the integral above), recompute the value of the integral in (4a).

(10 points)

5) Evaluate the following integrals

5a) $\displaystyle\int_{|z|=1} \frac{3}{(z-2)^4}\, dz$

(10 points)

5b) $\displaystyle\int_{|z|=3} \frac{3}{(z-2)^4}\, dz$

(10 points)

5c) $\displaystyle\int_{|z|=3} \frac{3\sin(z)}{(z-2)^4}\, dz$

(10 points)

6a) What is the radius of convergence for the power series

$$\mathrm{f}(z) = \sum_{n=1}^{\infty} \frac{(n+1)\, z^n}{n!}$$

(10 points)

6b) Differentiate and/or integrate term by term to find a closed form expression for $\mathrm{f}(z)$.

(10 points)

6b) Express $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$ in terms of e. (I remember a problem like this on the GRE practice exam).

(10 points)