Math 4200
Exam 1 solutions

1a) f analytic at $z_0 \in \mathbb{C}$ iff $\displaystyle\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} =: f'(z_0)$ exists.

b) Write $f(x+iy) = u(x,y) + i\,v(x,y)$ where $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$.

CR: $\boxed{\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}}$

f is analytic at z_0 iff $F(x,y) := \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is real differentiable at (x_0,y_0) and CR eqtns hold.

(Real differentiability at (x_0,y_0) is implied $u,v \in C^1$ in a nbhd of (x_0,y_0).)

c) $D_t\,(f(\gamma(t)) = f'(\gamma(t))\,\gamma'(t)$

d) $\partial_x f(x+iy) = f'(x+iy)\cdot \underbrace{1}_{\partial_x x}$

$\partial_y f(x+iy) = f'(x+iy)\, i \quad \Rightarrow f_y = i f_x$

$\Rightarrow u_y + i v_y = i(u_x + i v_x) = -v_x + i u_x$

$\boxed{\Rightarrow \begin{matrix} u_y = -v_x \\ v_y = u_x \end{matrix}}$

$\partial_r f(re^{i\theta}) = f'(re^{i\theta})\, e^{i\theta}$

$\partial_\theta f(re^{i\theta}) = f'(re^{i\theta})\, r i e^{i\theta}$

$\Rightarrow f_\theta = r i f_r \Rightarrow u_\theta + i v_\theta = ir(u_r + i v_r) = i r u_r - r v_r \Rightarrow \boxed{\begin{matrix} u_\theta = -r v_r \\ v_\theta = r u_r \end{matrix}}$

2) a) $\displaystyle\int_{|z|=2} \frac{1}{z}\,dz$ $\qquad z = 2e^{it} \quad 0 \le t \le 2\pi$, $\quad dz = 2ie^{it}\,dt$

$$:= \int_0^{2\pi} \frac{1}{2\cancel{e^{it}}}\, 2i\cancel{e^{it}}\,dt = \int_0^{2\pi} i\,dt = \boxed{2\pi i}$$

b) $e^{\sin z}$ is entire, $\mathbb{C}$ is simply connected, so $e^{\sin z}$ has an antiderivative on $\mathbb{C}$, γ is closed

so $\displaystyle\int_{|z|=2} e^{\sin z}\,dz = \boxed{0}$

c) $\displaystyle\int_{|z|=2} \bar{z}\,dz$ $\qquad z = 2e^{it}$, $\bar{z} = 2e^{-it}$, $\quad dz = 2ie^{it}\,dt$, so int $= \displaystyle\int_0^{2\pi} 2\cancel{e^{-it}}\, 2i\cancel{e^{it}}\,dt = (4i)(2\pi) = \boxed{8\pi i}$

d) $\displaystyle\int_{|z|=2} \frac{1}{|z|}\,dz = \int_{|z|=2} \frac{1}{2}\,dz = \frac{1}{2} z \Big]_2^2 = 0$ (FTC)

e) $\displaystyle\int_\gamma \frac{1}{|z|}\,|dz| = \int_{|z|=2} \frac{1}{2}\,|dz| = \frac{1}{2}\,(\text{circumfrence}) = \frac{1}{2}\cdot 2\pi\cdot 2 = \boxed{2\pi}$

arclength

3a) $f(z) = \sqrt{z^2+1}$

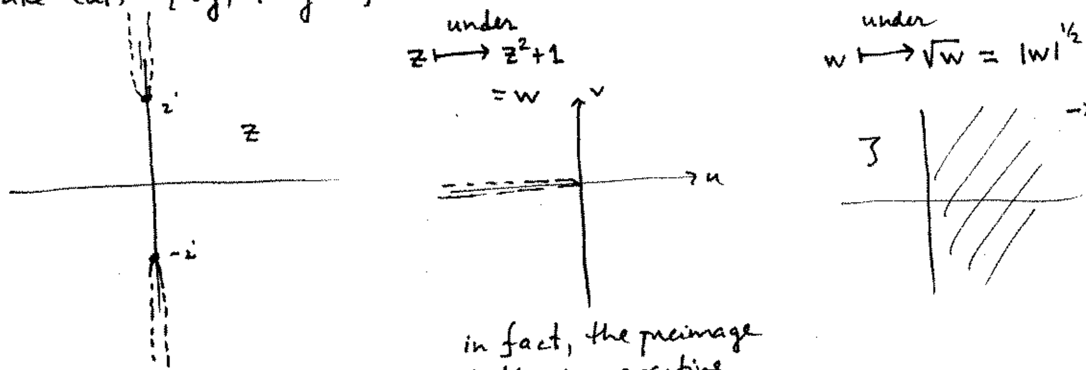
branch points $z = \pm i$

take cuts $\{iy,\ 1 \le |y| < \infty\} \longmapsto \{v \mid v \le 0\} \longmapsto \{\zeta \text{ s.t. } \operatorname{Re}\zeta > 0\}$

under $z \longmapsto z^2+1 = w$ under $w \longmapsto \sqrt{w} = |w|^{1/2} e^{i(\frac{1}{2}\arg w)}$

$-\pi < \arg w < \pi$

$= e^{\frac{1}{2}\log w}$ with standard branch of the argument



in fact, the preimage of the non-positive real axis is exactly the union of the 2 cuts in the z-plane, using Euler's formula.

Summary

$$f(z) = e^{\frac{1}{2}\log(z^2+1)} \quad \text{with standard branch of log}$$

3b)

$$f'(z) = e^{\frac{1}{2}\log(z^2+1)} \underbrace{\frac{\partial}{\partial z}\frac{1}{2}\log(z^2+1)}_{\frac{1}{2}\frac{1}{z^2+1}2z} \quad \text{(chain rule)}$$

$$= (z^2+1)^{1/2}\ \frac{1}{2}\frac{1}{z^2+1}2z$$

$$= \frac{z}{(z^2+1)^{1/2}}$$

4) For 4b) consult class notes or text.

For 4a):

$$\oint_{\partial R} f(z)\,dz = \int_{\partial R} (u+iv)(dx+idy) = \int_{\partial R} u\,dx - v\,dy + i\int_{\partial R} v\,dx + u\,dy \quad \left(\begin{matrix}\text{as checked in}\\ \text{class \& text}\end{matrix}\right)$$

$$= \iint_R (-v)_x - u_y\,dA + i\iint_R u_x - v_y\,dA \quad \text{Green's Thm !!}$$

but CR says $\begin{cases} v_x = -u_y \Rightarrow 1^{st} \text{ integrand} = 0 \\ u_x = v_y \Rightarrow 2^{nd} \text{ integrand} = 0 \end{cases}$

$= 0 + 0 = 0$ ■

5a) $A \subseteq \mathbb{C}$ is simply connected if it is connected and any (cont.) closed path $\gamma : [0,1] \to A$ is homotopic as a closed curve in A to some constant (point) map $\gamma_1 : [0,1] \to z_0 \in A$.

5b) If A is open & simply connected, and if $f : A \to \mathbb{C}$ is analytic on A then $\exists$ an antiderivative $F : A \to \mathbb{C}$, i.e. $F'(z) = f(z)$ $\forall z \in A$. (and F is unique up to an additive constant.)

5c) If $\mathbb{C}\setminus\{0\}$ was simply connected then the antiderivative theorem would hold, in particular for $f(z) = \frac{1}{z}$.

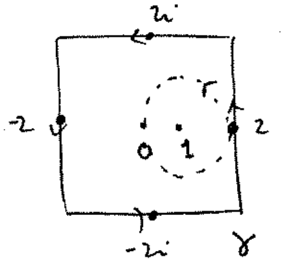
This would imply $\oint_{|z|=1} \frac{1}{z}\,dz = 0$ since the unit circle (oriented c.c.) is closed.

But, as we have frequently computed,

$$\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i \quad \Rightarrow\!\Leftarrow$$

Thus $\mathbb{C}\setminus\{0\}$ is not simply connected.

6)



$$\int_\gamma \frac{1}{z-1} + \frac{2}{z^2}\,dz = \underbrace{\int_\gamma \frac{1}{z-1}\,dz}_{} + \underbrace{\int_\gamma \frac{2}{z^2}\,dz}_{}$$

$$\int_\gamma \frac{1}{z-1}\,dz = \oint_{|z-1|=1} \frac{1}{z-1}\,dz = 2\pi i$$

$$\int_\gamma \frac{2}{z^2}\,dz = -\frac{2}{z}\Big]_2^2 = 0,$$ i.e. $-\frac{2}{z}$ is an antiderivative of $\frac{2}{z^2}$

(since $\displaystyle\int_{|z-a|=r} \frac{1}{z-a}\,dz = 2\pi i$)

because the circle $|z-1|=1$ is homotopic to the curve γ, (for example, one could use radial projection towards 1, write $\gamma_1(t) = 1 + \frac{\gamma(t)-1}{|\gamma(t)-1|}$ which is a (non-standard) parameterization of $|z-1|=1$, and

$$H(s,t) = (1-s)\gamma(t) + s\gamma_1(t) \quad 0 \le s \le 1$$

So, the final answer is $\boxed{2\pi i}$