EXAM 1

Math 4200-1 October 3, 2007

Each complete problem below is worth 30 points. Choose any three out of the six problems to do. You receive 10 points for your good name. If you try more than three problems, indicate clearly which three you want graded. This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

1a) Define what it means for a function f(z) to be complex differentiable at a point $z_0 \in C$.

(6 points)

1b) What are the Cauchy-Riemann equations? Explain precisely how they are related to complex differentiability (proofs not required).

(6 points)

1c) State the chain rule for curves, i.e. for the t-derivative of the composition $f(\gamma(t))$, where $\gamma(t)$ is a complex curve, and f is analytic at $\gamma(t)$.

(6 points)

1d) Use the chain rule for curves to derive the Cauchy Riemann equations in rectangular coordinates, i.e. by considering the composition of the analytic function f with coordinate curves $\gamma_1(x) = x + i\, y_0$ and $\gamma_2(y) = x_0 + i\, y$.

(6 points)

1e) Use an analogous argument to (1d) to derive the Cauchy-Riemann equations in polar coordinates.

(6 points)

2) Let γ be the circle $|z| = 2$, traversed once counterclockwise. Find the values of

2a) $\displaystyle\int_\gamma \frac{1}{z}\, dz$ 2b) $\displaystyle\int_\gamma e^{\sin(z)}\, dz$ 2c) $\displaystyle\int_\gamma \bar{z}\, dz$ 2d) $\displaystyle\int_\gamma \frac{1}{|z|}\, |dz|$ 2e) $\displaystyle\int_\gamma \frac{1}{|z|}\, dz$

(6 points each)

3a) Define a branch of the function

$$f(z) = (z^2 + 1)^{\left(\frac{1}{2}\right)}$$

on a suitable simply connected domain containing the unit disk D(0;1). Your domain should be the complex plane minus two rays. (Hint: make the domain star-shaped with respect to the origin.)

(20 points)

3b) Express f(z) as a composition using the complex logarithm function, and then use the chain rule to prove that $f'(z) = \dfrac{z}{(z^2 + 1)^{\left(\frac{1}{2}\right)}}$.

(10 points)

4) A key part of the proof of the deformation theorem is the rectangle lemma, which is used as a tool in constructing local antiderivatives to analytic functions. This lemma states that if a closed coordinate rectangle is contained in an open domain $A \subseteq C$, and if f(z) is analytic on A, then the contour integral around the boundary of the rectangle is zero. Do either (4a) or (4b) below - you only need to do ONE of these for full credit.

4a) If f(z) analytic and also C^1 on A, then use Green's Theorem to prove the rectangle lemma, after first writing the contour integral in terms of two line integrals.

OR

4b) Reproduce Goursat's beautiful proof of the rectangle lemma, which is valid under the weaker assumption that f(z) just be analytic on A.

(30 points)

5a) Define precisely what it means for an open connected domain $A \subseteq C$ to be simply connected.

(5 points)

5b) State the antiderivative theorem for simply connected domains $A \subseteq C$.

(5 points)

5c) Use the antiderivative theorem to prove that the complex plane complement the origin is NOT simply connected.

(20 points)

6) Let γ be the piecewise C^1 curve which is the boundary of the square $\{-2 \leq x \leq 2, -2 \leq y \leq 2\}$, oriented counterclockwise. Evaluate

$$\int_\gamma \frac{1}{z-1} + \frac{2}{z^2}\, dz.$$

If you use homotopies which are geometrically clear to evaluate pieces of this integral you need not display the explicit formulas.

(30 points)