Math 4200
Wednesday Dec. 5.

Magic function prep for Gary's presentation Friday on the Riemann zeta for prime number theorem, following "Complex Analysis" by Ahlfors.

Infinite products!

\[
G(z) := \prod_{n=1}^{\infty} \frac{1}{1 + \frac{1}{n^2}} e^{-\frac{z}{n}} = \left( \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^2}\right) e^{-\frac{z}{n}} \right) \lim_{N \to \infty} \left( \prod_{n=1}^{N} \left(1 + \frac{z}{n^2}\right) e^{-\frac{z}{n}} \right)
\]

\[e^{-\sum_{n=1}^{\infty} \frac{\log(1 + \frac{z}{n^2}) - \frac{z}{n}}{n}} \quad (|z| < 1)
\]

Thus \(G(z)\) has simple zeroes at each negative integer.

In fact:

\[
\frac{\sin \pi z}{\pi} = \pi G(z) G(-z)
\]

Check! \(\frac{\pi^2 G(z) G(-z)}{\sin \pi z} = F(z)\) is entire, no zeroes, so \(F(z) = e^f(z)\) where \(f(z) := \log F(z) + \int_{\frac{1}{2}}^{z} \frac{F(t)}{F'(t)} dt \)

\(\pi \cot \pi z + f'(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{1 + \frac{z}{n^2}} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{1 - \frac{z}{n^2}} \right)
\]

\[\pi \cot \pi z + f'(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{1 + \frac{z}{n^2}} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{1 - \frac{z}{n^2}} \right)
\]

\[\Rightarrow f''(z) = 0 \quad \Rightarrow F(z) \text{ const}; \quad \lim_{z \to 0} F(z) = 1 \quad \square\]
Back to \( G(z) \):

\[ G(z-1) \text{ as zero (simple)} \text{ at } z-1 \text{ = neg integer} \]
\[ z = 0, \text{ or } z = \text{neg. integer}. \]

Thus
\[ G(z-1) = e^{\gamma} z G(z) \quad \text{where } \gamma(z) \text{ is entire. (See page 1 !)} \]

In fact, \( \gamma(z) \) is a constant, called Euler's constant \( \gamma \)

\[ \text{check: log diff:} \]
\[ \sum_{n=1}^{\infty} \left( \frac{1}{2-1+n} - \frac{1}{n} \right) = \gamma' + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n} \]
\[ \gamma' = \gamma' + \frac{1}{z} \quad ; \quad \gamma(z) = \text{const} \]
\[ \gamma = \gamma(1) \]

But \( G(1) = 1 \)
\[ \therefore G(z) = \frac{1}{z} e^{\gamma(1)} \]

\[ \therefore \quad e^{\gamma} = G(1) = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}} \]
\[ \Rightarrow \quad e^{\gamma} = G(1) = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}} \]
\[ \therefore \quad \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n} \right) \]
\[ \approx 0.57722 \]

\[ T(1) = \frac{e^{\gamma}}{G(1)} = 1 \quad \Rightarrow \quad T(z) = z T(z-1) = z! \]

\[ T(2) = 2 \cdot 1 = 2! \]
\[ T(3) = 3 \cdot 2 = 3! \]
\[ \vdots \]
\[ T(n) = n! = (n-1)! \]

\[ T \] is the factorial function.

Also

\[ T(z) T(1-z) = T(z) (-z) T(-z) = (-z) \left( \frac{1}{z} \right) \left( \frac{1}{G(z) G(-z)} \right) = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}} \]

Another formula for \( T(z) \), valid for \( Re z > 0 \):

\[ T'(z) = \int_{0}^{\infty} e^{-z t} t^{z-1} dt \]

\[ \left( T(z) = \frac{1}{z} e^{\gamma(1)} \right) \quad \text{trouble is at } t = 0, \text{ not } t = \infty \)