Math 4200
Wednesday Dec. 5.

Magic function prep for Gary's presentation Friday on the Riemann zeta fcn & prime number theorem.
following "Complex Analysis" by Ahlfors

Infinite products!
Let

$$\boxed{G(z) := \prod_{n=1}^{\infty} (1+z/n)e^{-z/n}} = \left(\prod_{n=1}^{n_1}(1+z/n)e^{-z/n}\right) \lim_{N\to\infty} \underbrace{\left(\prod_{n=n_1}^{N}(1+z/n)e^{-z/n}\right)}_{e^{\sum_{n_1}^{N}\log(1+z/n)-z/n}} \qquad (|z/n_1|<1)$$

$$\log(1+z/n) - z/n$$
$$= \cancel{z/n} - \tfrac{1}{2}(z/n)^2 + \tfrac{1}{3}(z/n)^3 - \dots - \cancel{z/n}$$
$$|\log(1+z/n) - z/n| \le |z/n|^2 M \quad \text{for } |z/n|<1/2$$

so corrected log series converges uniformly absolutely on bounded sets of $\mathbb{C}$, so

$$G(z) = \lim_{N\to\infty}\prod_{n=1}^{N}(1+z/n)e^{-z/n}$$

converges unif. on compact subsets $\mathbb{C}$

$G(z)$ has simple zeroes at each negative integer.

Thus

$z\,G(z)G(-z)$ has simple zeroes at each integer.

In fact:

$$\boxed{\frac{\sin\pi z}{\pi} = z\,G(z)G(-z)}$$

Check! $\dfrac{\pi z\,G(z)G(-z)}{\sin\pi z} = F(z)$ is entire, no zeroes, so
$F(z) = e^{f(z)}$

where $f(z) := \log F(z_0) + \displaystyle\int_{z_0}^{z} \frac{F'(\zeta)}{F(\zeta)}\,d\zeta$

pf: $\left(e^{-f(z)}F(z)\right)' = e^{-f}\left[-\frac{F'}{F}F + F\right] = 0$
& $e^{-f(z_0)}F(z_0) = 1$ ∎

$$z\,G(z)G(-z) = \frac{\sin\pi z}{\pi}e^{f(z)}$$

logarithmically differentiate:

$$\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1/n}{1+z/n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty}\left(\frac{-1/n}{1-z/n} + \frac{1}{n}\right)$$
$$= \pi\frac{\cos\pi z}{\sin\pi z} + f'(z)$$

i.e.

$$\pi\cot\pi z + f'(z) = \underbrace{\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty}\frac{1}{z-n} + \frac{1}{n}}_{\pi\cot\pi z\,!}$$

$\Rightarrow f'(z) \equiv 0$; $\Rightarrow F(z)$ const; $\lim_{z\to 0} F(z) = 1$ ∎

Back to $G(z)$:

$G(z-1)$ as zeros (simple) at $z-1 = $ neg integer
$z = 0$, or $z = $ neg. integer.

Thus
$$G(z-1) = e^{\gamma(z)} z\, G(z) \qquad \text{where } \gamma(z) \text{ is entire. (See page 1!)}$$

In fact, $\gamma(z)$ is a constant, called Euler's constant γ

check: log diff:
$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \gamma'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n}$$
$$\frac{1}{z+(n-1)}$$
$$\cancel{\frac{1}{z}} = \gamma'(z) + \cancel{\frac{1}{z}} \;; \quad \gamma(z) \equiv \text{const}$$
$$\gamma = \gamma(1)$$

$$G(z-1) = e^{\gamma} z\, G(z)$$

But $G(0) = 1$

& $G(1-1) = e^{\gamma(1)}\, 1\, G(1)$

$$\Rightarrow \underline{e^{-\gamma} = G(1)} = \underbrace{\prod_{n=1}^{\infty}(1+1/n)e^{-1/n}}$$
$$\prod_{n=1}^{N}(\;) = (N+1)e^{-\sum_{n=1}^{N}\frac{1}{n}}$$
$$\Rightarrow \gamma = \lim_{n\to\infty}\left(1 + 1/2 + 1/3 + \dots + 1/n - \ln n\right)$$
$$\approx .57722$$

so $H(z) = e^{\gamma z} G(z)$ satisfies
$$H(z-1) = z\, H(z)$$

So $\Gamma(z) = \dfrac{1}{z\,H(z)}$ satisfies

$$\boxed{\begin{aligned} &\Gamma(z+1) = z\,\Gamma(z) \\ &\Gamma(z) := \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty}(1 + z/n)^{-1} e^{z/n} \end{aligned}} = \boxed{\frac{e^{-\gamma z}}{z\, G(z)}}$$

The Γ fcn $\to$ no zeros.
simple poles at non-positive integers

$$\Gamma(1) = \frac{e^{-\gamma}}{G(1)} = 1 \;\Rightarrow\; \Gamma(2) = 1\,\Gamma(1) = 1$$
$$\Gamma(3) = 2\cdot 1 = 2!$$
$$\Gamma(4) = 3\cdot 2! = 3!$$
$$\vdots$$
$$\Gamma(n) = (n-1)!$$

Γ is the factorial function!

also

$$\boxed{\Gamma(z)\Gamma(1-z) = \Gamma(z)(-z)\Gamma(-z)} = (\cancel{-z})\left(\frac{1}{z(\cancel{-z})}\right)\left(\frac{1}{G(z)G(-z)}\right) = \frac{\pi}{\sin(\pi z)}$$

$$\left(\Gamma(\tfrac{1}{2})^2 = \pi,\; -\tfrac{1}{2}! = \sqrt{\pi}\ ?!\right)$$

Another formula for $\Gamma(z)$, valid for $\mathrm{Re}\, z > 0$:

$$\boxed{\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt} \quad \left(= \int_0^{\infty} e^{-t} \underbrace{e^{(z-1)\ln t}}_{t^{x-1} e^{iy\ln t}} dt \;, \text{ trouble is at } t=0, \text{ not } t=\infty \right)$$