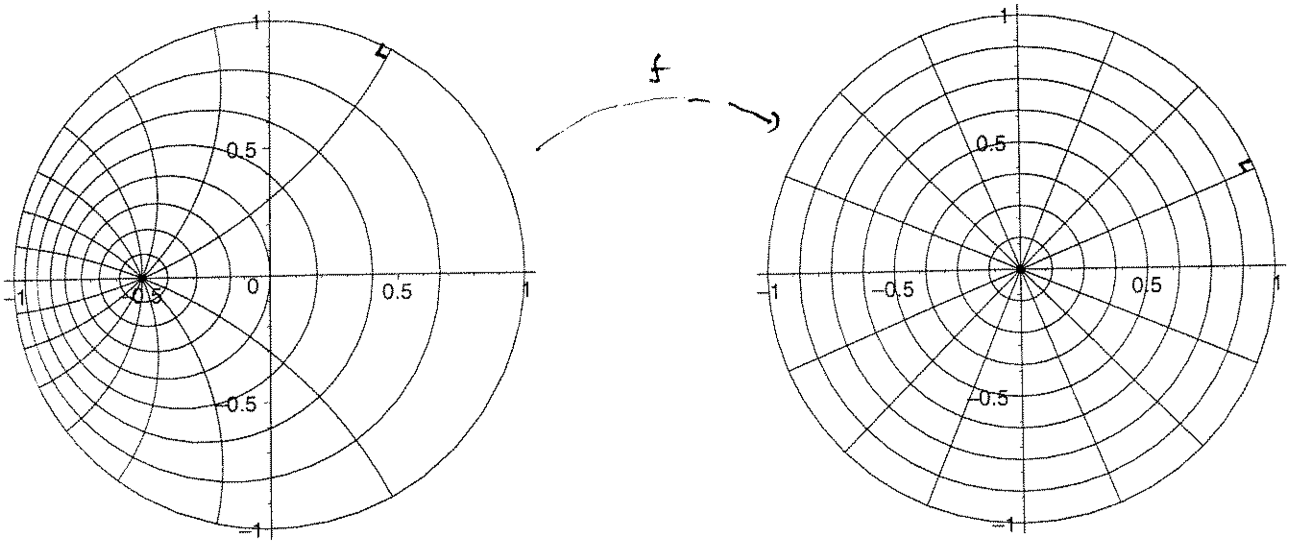
Math 4200
Monday Dec. 3.

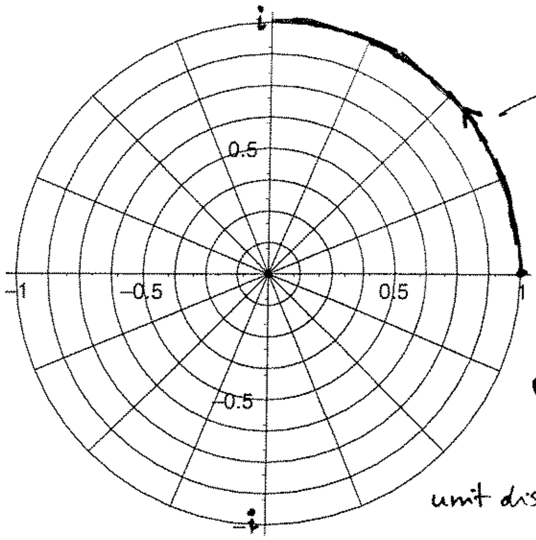
- Mobius transformations (page 2 Friday)

<u>Example</u> $f(z) = \dfrac{z + 1/2}{1 + \frac{1}{2} z}$:



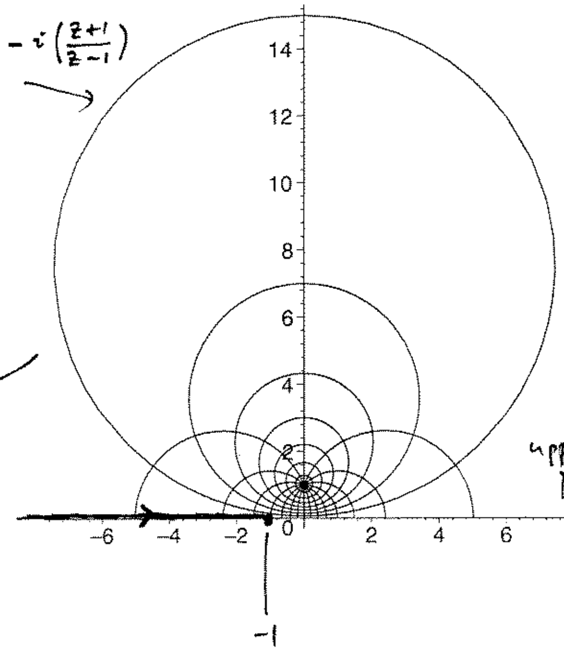
- Applications of conformal mapping to harmonic functions. Recall, harmonic $\circ$ analytic = harmonic. (Why?)

Use Friday FLT as example



$f(z) = -i\left(\frac{z+1}{z-1}\right)$

$g(w) = \frac{w-i}{w+i}$



$0 \longleftrightarrow i$

$1 \longleftrightarrow \infty$

$-1 \longleftrightarrow 0$

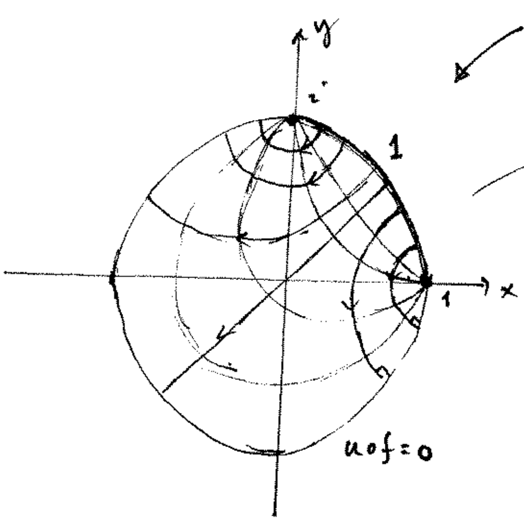
$-i \longleftrightarrow 1$

$i \longleftrightarrow -1$

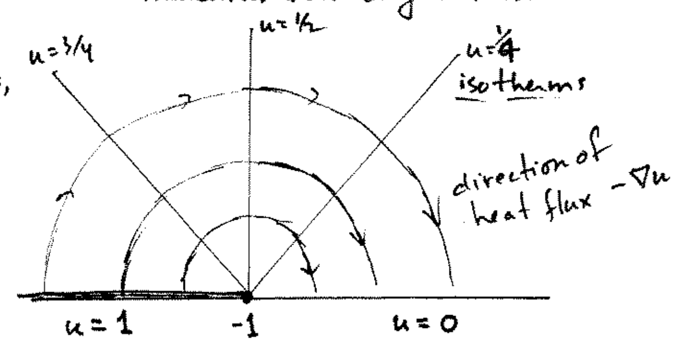
Consider $u(w) = \frac{\arg(w-(-1))}{\pi} = \frac{\arg(w+1)}{\pi}$

Is harmonic, with indicated boundary values:

Thus, $u \circ f(z)$ solves harmonic fcn with corresponding boundary temperatures, isotherms & temp grads



$f(z)$



other applications: fluid mechanics, air foils.

In <u>Riemannian Geometry</u> one studies manifolds M which have (infinitesimal) elements of arc-length, so that one can study lengths of curves. (which leads to a lot more).

An <u>isometry</u> ~~preserves~~ $T: M \to M$ preserves all curve lengths.

<u>example</u> $\mathbb{R}^2$, $ds = \sqrt{dx^2 + dy^2}$ (so $\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}\, dt$).

<u>Theorem</u>: The only orientation preserving isometries of $\mathbb{R}^2$ are

$$T(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x} + \vec{b} \quad = T_{\theta,b}(\vec{x})$$

"proof": let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be any isometry. (which preserves orientation)

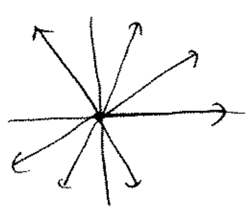
Since T preserves all curve lengths, by looking at short curves we deduce it preserves arclength elements, (by taking a limit)

2270 {so the derivative matrix of T at any point must be a rotation matrix (but, a prior, this rotation could change as you change pts)

Compose T with some $T_{\theta,b}$ so that

$$\left.\begin{array}{l} (T \circ T_{\theta,b})(0) = 0 \\ (\qquad)'(0) = I \end{array}\right\} \; T \circ T_{\theta,b} \text{ is isom which fixes origin \& tangent vector at origin.}$$

T an isometry $\Rightarrow$ shortest lines (paths) btwn pts $\to$ shortest lines (paths) btwn pts.



$\Rightarrow$ pos. x-axis $\to$ a curve which minimizes dist btwn pts on it, with
$0 \to 0$
x-dir $\to$ x-dir

$\Rightarrow$ pos x-axis $\to$ pos x-axis, as identity map

$\Rightarrow$ each ray from origin $\to$ itself, as identity!

$\Rightarrow T \circ T_{\theta,b} = id$

$\Rightarrow T = T_{\theta,b}^{-1}$ ∎

<u>example</u> $S^2 \subset \mathbb{R}^3$

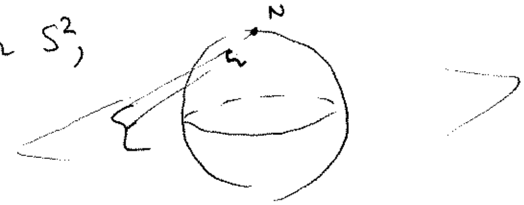
$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } x^2 + y^2 + z^2 = 1 \right\}$$

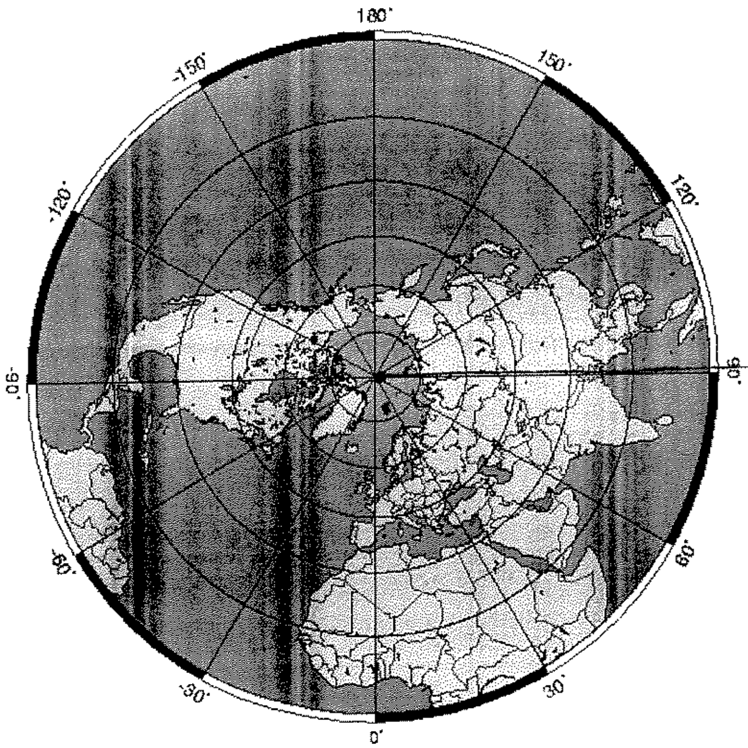
by a completely analogous argument, one can show that the only orient. pres. isoms of S^2 are the 3×3 rotation matrices (called $S.O.(3)$)

(i.e. compose with such a rot. matrix, to fix north pole & directions at north pole.).

By the way, using a stereographic chart for S^2,

$$\boxed{ds = \left(\frac{2}{1 + x^2 + y^2}\right)\sqrt{dx^2 + dy^2}}$$





<u>Exercise</u> $(R=1)$

using
$$ds = \frac{2}{1+x^2+y^2}\sqrt{dx^2+dy^2}$$
compute the "length" of the positive x-axis.

You <u>should</u> get π!
(why?)

There is one more model space !!

$\mathbb{H}^2$ = upper half space (Hyperbolic space) Lobachevsky

$$ds = \frac{1}{y}\sqrt{dx^2+dy^2}$$

and the isometries are

$$T(z) = \frac{az+b}{cz+d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$$

det = 1

a, b, c, d real

(these correspond to Möbius transforms in unit disk)

generated by $T_1(z) = \lambda z$, $\lambda > 0$ $\left(\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \right)$

$T_2(z) = z + b$ $b \in \mathbb{R}$ $\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right)$

$T_3(z) = -\frac{1}{z}$ $\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$

"straight forward" to show T_1, T_2, T_3 are isoms.

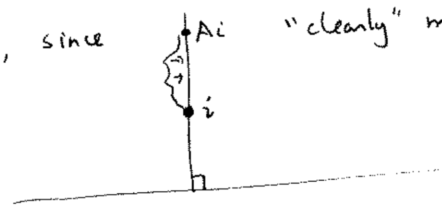
every orientation preserving isom must be conformal $\Rightarrow$ These are all.

e.g. $(T_1 \circ \gamma)'(t) = \lambda \gamma'(t)$

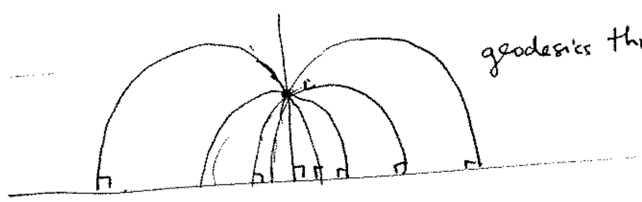
if $\gamma(t) = x_0 + iy_0$

$$\frac{ds_{\mathbb{H}}}{dt} = \|\gamma'(t)\|_{\mathbb{H}} = \frac{\|\gamma'(t)\|_{\mathbb{E}}}{y_0} \qquad \|(T_1 \circ \gamma)'(t)\|_{\mathbb{H}} = \frac{\|\lambda \gamma'(t)\|_{\mathbb{E}}}{\lambda y_0} = \|\gamma'(t)\|_{\mathbb{H}}.$$

Then, since



"clearly" minimizes length btwn i & Ai, all its images under isoms are also "geodesics"

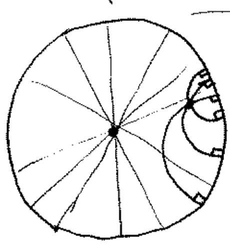


geodesics thru i

another model (chart) for this space is the Poincaré disk $\mathbb{D} = \{z \mid |z| < 1\}$

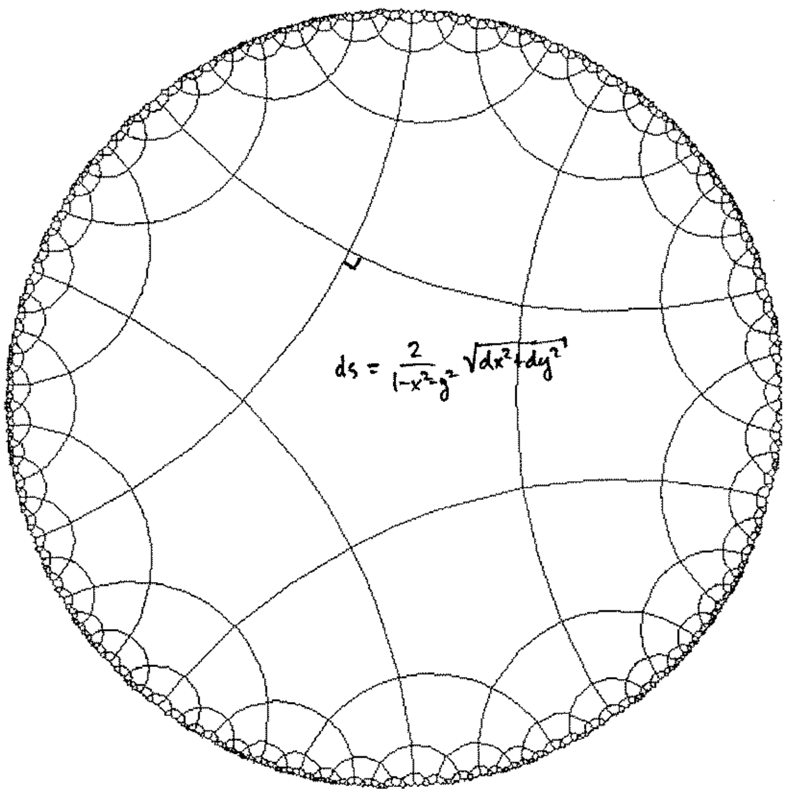
$$\boxed{ds = \frac{2}{1-x^2-y^2}\sqrt{dx^2+dy^2}}$$

(Compare to ds for sphere chart!)



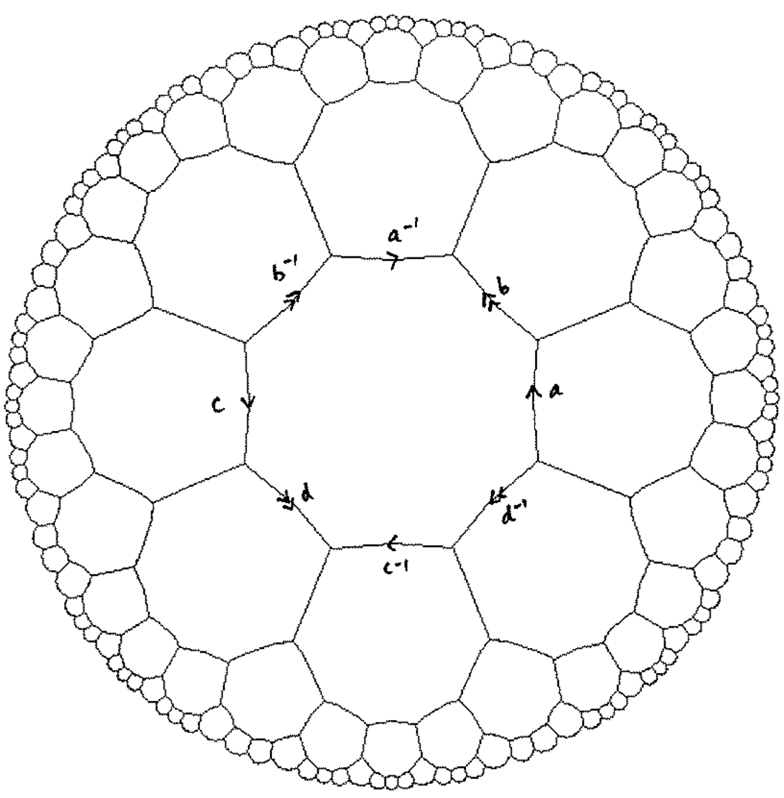
and the page 1 $f(z), g(w)$ are isometries between $\mathbb{H}^2$ & $\mathbb{D}^2$.

(the isometries of $\mathbb{D}$ are the Möbius transformations)



unit disk
map of the
hyperbolic plane.

isometric, $\pi/2$ vertex-angled pentagons



equilateral octagons with interior angles $2\pi/3$.

Identification of sides leads to a genus 2 surface (doughnut w 2 holes), of constant curvature.