Math 4200
Fri Aug 31

HW for Fri Sept 7
§1.5 1ad, 3b, 5c, 6b
(in 5c, 6d describe what the "derivative" or "differential" map does to tangent vectors based at z_0)
8, 9, 10, 11, 13, (then look at 14, but don't hand in 14)
16, 18abc, 19

§1.5 Complex differentiability

Let $f: A \to \mathbb{C}$, $A \subset \mathbb{C}$, A open.

We never actually wrote

Def $\lim_{z \to z_0} f(z) = L$ iff $\boxed{}$

(don't forget that $z \neq z_0$)

or this version of the basic limit thms:

Thm If $\lim_{z\to z_0} f(z) = L$, $\lim_{z\to z_0} g(z) = M$ then

(a) $\lim_{z \to z_0} f(z) + g(z) = L + M$

(b) $\lim_{z\to z_0} f(z) g(z) = LM$

(c) $\lim_{z\to z_0} f(z)/g(z) = L/M$ if $M \neq 0$.

(on your own you should check that you can prove these)

We did write

Def f is continuous at z_0 iff $\lim_{z\to z_0} f(z) = f(z_0)$

Since $f(z) = z$ is cont, and $g(z) = c$ is too, we deduce from the limit theorem that all polynomials are continuous, and rational fns are cont except where denom. = 0.

(in fact, from §1.4 HW, $f(z) = f(x+iy) = u(x,y) + i v(x,y)$ is continuous at a point iff u, v are.)

Notational lemma: The following limit statements are all equivalent

(1) $\lim_{z\to z_0} f(z) = L$ (2) $\lim_{h \to 0} f(z_0 + h) = L$ (3) $\lim_{z \to z_0} |f(z) - L| = 0$

(4) $\lim_{h\to 0} |f(z_0+h) - L| = 0$

pf (1)$\Leftrightarrow$(2): relate h to z by $z_0 + h = z$ in the respective def's. (then $z \to z_0$ iff $h \to 0$)

(3)$\Leftrightarrow$(4) "

(1)$\Leftrightarrow$(3): (1) iff $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

(3) iff $\forall \varepsilon>0\ \exists \delta > 0$ s.t. $0 < |z-z_0| < \delta \Rightarrow \big|\, |f(z) - L| - 0 \,\big| < \varepsilon$

identical!

<u>Def</u> $f: \overset{\mathbb{C}}{\underset{\cup}{A}} \to \mathbb{C}$, A open, $z_0 \in A$.

f is <u>complex differentiable</u> or <u>analytic</u> at z_0, with <u>derivative</u> $f'(z_0)$ iff

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) \qquad \left(= \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} \right)$$

<u>Example</u>: $f'(z^n) = nz^{n-1}$ $\quad n \in \mathbb{N}$

<u>pf</u>: $$\lim_{h\to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h \to 0} \frac{\cancel{z^n} + \binom{n}{1} z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \dots + \binom{n}{n-1} z h^{n-1} + h^n - \cancel{z^n}}{h}$$

$$= \lim_{h \to 0} nz^{n-1} + h(\underset{\substack{\uparrow \\ \text{poly in} \\ h \,\&\, z}}{\text{stuff}}) = nz^{n-1}$$

<u>Theorem</u> If f, g are analytic at z_0 then so are $f+g$, fg. If also $g(z_0) \neq 0$, then f/g is analytic at z_0.

Furthermore

(i) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

(ii) $(cf)'(z_0) = c f'(z_0)$ $\quad (c$ const$)$

(iii) $(fg)'(z_0) = (f'g + fg')(z_0)$

(iv) $(f/g)'(z_0) = \dfrac{f'g - fg'}{g^2}(z_0)$

pf of either (iii) or (iv):

<u>Cor</u>: You can differentiate any rational function of z as if you were taking Calculus.

<u>Example</u> $f(z) = |z|$ is <u>not</u> complex differentiable anywhere in $\mathbb{C}$! (See HW set)

In fact, if $u(x,y)$, $v(x,y)$ are randomly chosen real C^1 functions

$f(x+iy) := u(x,y) + iv(x,y)$ is almost guaranteed to <u>NOT</u> be complex differentiable

<u>Example</u> $f(z) = \bar{z}$ is <u>NOT</u> complex differentiable.

Example $f(z) = e^z$ is analytic on $\mathbb{C}$, with $f'(z) = e^z$ hurray!

proof $$\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h \to 0} e^z \left(\frac{e^h - 1}{h} \right) = e^z \underbrace{\lim_{h\to 0} \frac{e^h-1}{h}}_{\text{so if this limit is } 1, \text{ claim is true.}}$$

write $h = h_1 + i h_2$
and try to show $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

f.y.i.
Theorems we'll prove next time:

Chain rule if f is differentiable at z_0
and g is " " $f(z_0)$
Then $g \circ f$ is differentiable at z_0, and $(g \circ f)'(z_0) = g'(f(z_0)) f'(z_0)$

Cor $\sin z, \cos z$, etc. are diff'ble (with "same" derivatives you learned in Calculus)

Inverse function theorem: If $f(z)$ is analytic in a neighborhood of z_0,
with $f'(z_0) \neq 0$, and if $u(x,y) := \mathrm{Re}\, f(z)$, $v(x,y) := \mathrm{Im}\, f(z)$ have continuous partial derivs in a nbhd of (x_0, y_0) ($u, v \in C^1$)

Then f is locally bijective, and
$\exists\, f^{-1}$ from a nbhd of $f(z_0)$ to a nbhd of z_0.
Furthermore f^{-1} is analytic in a nbhd of $f(z_0)$, and
$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}$$

Cor $\log z$ is analytic for $z \neq 0$, for any branch choice with z in the interior.
(and $\log'(e^w) = \frac{1}{e^w}$, i.e. $\log'(z) = \frac{1}{z}$.)

Cauchy-Riemann equations (related to the geometry of analytic functions ...)

Write $f(\underbrace{x+iy}_{z}) = u(x,y) + i\,v(x,y)$ $\quad u,v$ real

Define $f_x := \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x + iy) - f(x+iy)}{\Delta x} = u_x + i v_x$ (usual partial derivs of u,v)

$f_y := \lim_{\Delta y \to 0} \dfrac{f(z+i\Delta y) - f(z)}{\Delta y} = u_y + i v_y$

If $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists, then the restricted limits also exist:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta x \text{ real}}} \frac{f(z+\Delta x) - f(z)}{\Delta x} = f'(z)$$

$$\lim_{\substack{i\Delta y \to 0 \\ \Delta y \text{ real}}} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} = f'(z)$$

Thus $f_x = f'(z)$

$\frac{1}{i} f_y = f'(z)$

in particular,

$\boxed{f_y = i f_x}$, so (equating real & imag. parts)

$u_y + i v_y = i(u_x + i v_x)$

$$\boxed{\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \quad \text{Cauchy-Riemann eqtns} \quad CR}$$

so analytic $\Rightarrow$ CR

in fact, $\boxed{\begin{aligned} &CR + u,v \in C^1 \Rightarrow \text{analytic, } \underline{\text{and}} \\ &\qquad f'(z) = u_x + i v_x = f_x \\ &\qquad\qquad = -i u_y + v_y = -i f_y \end{aligned}}$

Example Use boxed claim above to reprove e^z is analytic, with $(e^z)' = e^z$