Math 4200
Wed Aug 29

§1.4 cont'd: adding functions to the mix of sets, sequences.

We'll be interested in

$$f: \underset{\mathbb{C}}{\overset{\cap}{A}} \to \mathbb{C}, \qquad \gamma: \underset{\mathbb{R}}{\overset{\cap}{I}} \to \mathbb{C} \qquad u: \underset{\mathbb{C}}{\overset{\cap}{A}} \to \mathbb{R}$$

complex fcns curves real valued functions

Since continuity is easily discussed for $F: \mathbb{R}^n \to \mathbb{R}^m$, we will temporarily adopt that generality, and use $B(x_0;\varepsilon) := \{x \in \mathbb{R}^n \text{ s.t. } \|x - x_0\| < \varepsilon\}$ notation (rather than disk notation $D(z_0;\varepsilon)$)

<u>Def</u> $F: \underset{\mathbb{R}^n}{\overset{\cap}{A}} \to \mathbb{R}^m$; For $B \subset A$ $f(B) := \{y \in \mathbb{R}^m \text{ s.t. } y = f(x) \text{ for some } x \in B\}$
For $B \subset \mathbb{R}^m$, $f^{-1}(B) := \{x \in A \text{ s.t. } f(x) \in B\}$

<u>Def</u> $F: \underset{x_0}{\overset{\in}{A}} \to \mathbb{R}^m$. <u>F is continuous at x_0</u> iff $\boxed{}$

<u>Def</u> F is continuous on A iff F is cont. at x_0, $\forall x_0 \in A$

<u>Def</u> $U \subset A$ is <u>relatively open</u> (w.r.t. A) iff $U = V \cap A$, $V \subset \mathbb{R}^n$ open
$W \subset A$ " <u>closed</u> iff $W = V \cap A$, V closed.

<u>Theorem</u>: The following are equivalent, for $x_0 \in A$, $F: \underset{\mathbb{R}^n}{\overset{\cap}{A}} \to \mathbb{R}^m$

(1) F is continuous at x_0
(2) F is sequentially continuous at x_0, i.e. $\forall \{x_n\} \subset A$, such that $\{x_n\} \to x_0$ then $\{f(x_n)\} \to f(x_0)$

(see Hw §1.4 #18)

<u>Theorem</u>: The following are equivalent, for $F: \underset{\mathbb{R}^n}{\overset{\cap}{A}} \to \mathbb{R}^m$

(1) F is continuous on A (ε-δ def.)
(2) $F^{-1}(\mathcal{O})$ is (relatively) open in A, $\forall \mathcal{O}$ open in $\mathbb{R}^m$

this is an important and useful characterization of continuous.
there's room on next page for proof.

<u>Theorem</u>: If the range of F, G is $\mathbb{C}$ $(\cong \mathbb{R}^2)$, and F, G is written in complex form

and if F, G have common domain $A \ni x_0$.

Then F, G continuous at x_0 implies

(1) $F+G$ cont at x_0

(2) FG cont at x_0

(3) F/G cont at x_0 if $G(x_0) \neq 0$

<u>Continuous functions and compactness</u>:

(1) <u>Thm</u>: If $K \subset \mathbb{R}^n$ is compact, $F: K \to \mathbb{R}^m$ continuous

Then $F(K) \subset \mathbb{R}^m$ is compact

(2) <u>Corollary</u>: If $K \subset \mathbb{R}^n$ is compact, $F: K \longrightarrow \mathbb{R}^1$

then F attains its minimum (infimum) "m"
& maximum (supremum) "M"

i.e. $\exists\, x_1 \in K$ s.t. $F(x_1) = \inf_{x \in K} F(x)$

$\exists\, x_2 \in K$ s.t. $F(x_2) = \sup_{x \in K} F(x)$

<u>Def</u>: $F: \overset{\substack{\mathbb{R}^n \\ \cup}}{A} \to \mathbb{R}^m$ is <u>uniformly continuous</u> iff

(3) <u>Theorem</u>: $\underset{\substack{\cap \\ \mathbb{R}^n}}{K}$ compact, $f: K \to \mathbb{R}^m$ continuous

$\Rightarrow f$ uniformly continuous

<u>Connectivity and functions</u>

(1) <u>Thm.</u> $\underset{\substack{\cap \\ \mathbb{R}^n}}{A}$ connected, $F: A \to \mathbb{R}^m$ continuous $\Rightarrow F(A)$ connected

<u>Def</u> $\overset{\substack{\mathbb{R}^n \\ \cup}}{A}$ is path connected iff $\forall x, y \in A \;\exists\; \gamma: [a,b] \to A$ continuous (a "path")

s.t. $\gamma(a) = x$, $\gamma(b) = y$

(2) <u>Thm</u>

A path connected $\Rightarrow$ A connected

(3) <u>Thm</u> A open, connected $\Rightarrow$ A path connected

final theorem today (really about compactness and functions)

<u>Thm</u> (positive distance theorem).

Let K compact, $\mathcal{O}$ open, $K \subset \mathcal{O}$

Then $\exists\, \delta > 0$ s.t.

$$\forall x \in K, \quad B(x;\delta) \subset \mathcal{O}.$$