

Math 4200
Wed Aug 29

(1)

§1.4 cont'd: adding functions to the mix of sets, sequences.

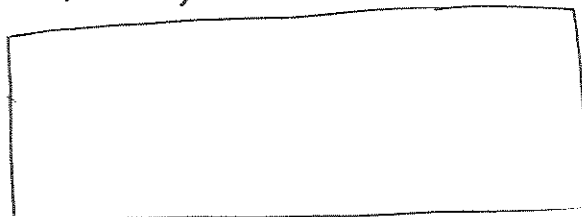
We'll be interested in

$$\begin{array}{ccc} f: A \rightarrow \mathbb{C} & , & \gamma: I \rightarrow \mathbb{C} & & u: A \rightarrow \mathbb{R} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{C} & & \mathbb{R} & & \mathbb{C} \\ \text{complex fun} & & \text{curves} & & \text{real valued functions} \end{array}$$

Since continuity is easily discussed for $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$, we will temporarily adopt that generality, and use $B(x_0, \varepsilon) := \{x \in \mathbb{R}^n \text{ s.t. } \|x - x_0\| < \varepsilon\}$ notation (rather than disk notation $D(x_0, \varepsilon)$)

Def $F: A \rightarrow \mathbb{R}^m$; For $B \subset A$ $f(B) := \{y \in \mathbb{R}^m \text{ s.t. } y = f(x) \text{ for some } x \in B\}$
 $\mathbb{R}^m$ For $B \subset \mathbb{R}^m$, $f^{-1}(B) := \{x \in A \text{ s.t. } f(x) \in B\}$

Def $F: A \rightarrow \mathbb{R}^m$. F is continuous at x_0 iff



Def F is continuous on A iff F is cont. at x_0 , $\forall x_0 \in A$

Def $U \subset A$ is relatively open (w.r.t. A) iff $U = V \cap A$, $V \subset \mathbb{R}^n$ open
 $W \subset A$ is relatively closed iff $W = V \cap A$, V closed.

Theorem: The following are equivalent, for $x_0 \in A$, $F: A \rightarrow \mathbb{R}^m$

- (1) F is continuous at x_0
 - (2) F is sequentially continuous at x_0 , i.e. $\forall \{x_n\} \subset A$, $\{x_n\} \rightarrow x_0$ such that then $\{f(x_n)\} \rightarrow f(x_0)$
- (see lhw §1.4 #18)

Theorem: The following are equivalent, for $F: A \rightarrow \mathbb{R}^m$

- (1) F is continuous on A (ε - δ def.)
- (2) $F^{-1}(O)$ is (relatively) open in A , $\forall O$ open in $\mathbb{R}^m$

this is an important and useful characterization of continuous.

there's room on next page for proof.

Theorem: If the range of $F|_{G'}$ is $\mathbb{C}$ ($\cong \mathbb{R}^2$), and $F|_G$ is written in complex form

and if F, G have common domain $A \subseteq X$.
Then F, G continuous at x_0 implies

- (1) $F+G$ cont at x_0
- (2) FG cont at x_0
- (3) F/G cont at x_0 if $G(x_0) \neq 0$

Continuous functions and compactness:

(1) Thm: If K is compact, $F: K \rightarrow \mathbb{R}^m$ continuous
Then $F(K)$ is compact

(2) Corollary: If K is compact, $F: K \rightarrow \mathbb{R}^1$
 $\bigcap_{\mathbb{R}^n}$ then F attains its minimum (infimum) "m"
 $\bigcup$ maximum (supremum) "M"
 i.e. $\exists x_1 \in K$ s.t. $F(x_1) = \inf_{x \in K} F(x)$
 $\exists x_2 \in K$ s.t. $F(x_2) = \sup_{x \in K} F(x)$

Def: $F: \overset{\mathbb{R}^n}{A} \rightarrow \mathbb{R}^m$ is uniformly continuous iff

(3) Theorem: $\underset{\mathbb{R}^n}{K}$ compact, $f: K \rightarrow \mathbb{R}^m$
continuous
 $\Rightarrow f$ uniformly continuous

Connectivity and functions

(1) Thm. $\underset{\mathbb{R}^n}{A}$ connected, $F: A \rightarrow \mathbb{R}^m$ continuous $\Rightarrow F(A)$ connected

Def $\overset{\mathbb{R}^n}{A}$ is path connected iff $\forall x, y \in A \exists \gamma: [a, b] \rightarrow A$ continuous
(a "path")
(2) Thm A path connected $\Rightarrow A$ connected
s.t. $\gamma(a) = x, \gamma(b) = y$

(3) Thm A open, connected $\Rightarrow A$ path connected

final theorem today (really about compactness and functions)

Thm (positive distance theorem).

Let K compact, $\mathcal{O}$ open, $K \subset \mathcal{O}$

Then $\exists \delta > 0$ s.t.

$$\forall x \in K, B(x; \delta) \subset \mathcal{O}.$$

