Math 4200
Monday Aug 27

§1.4 : 3220 material we need in this course : today: sets & sequences
(and the notation we will use) Wed: add functions to discussion

Sets

$D(z_0;r) = \{z \in \mathbb{C} \text{ s.t. } |z-z_0|<r\}$ "open disk", "r neighborhood" $(r>0)$

$D(z_0;r) \setminus \{z_0\} = \{z \in \mathbb{C} \text{ s.t. } 0<|z-z_0|<r\}$ "deleted open disk", "deleted r-neighborhood" $(r>0)$

$A \subset \mathbb{C}$ is a neighborhood of z_0 iff
$\exists\, r>0$ s.t. $D(z_0;r) \subset A$

$A \subset \mathbb{C}$ is open iff $\forall z_0 \in A\ \exists\, r>0$ s.t. $D(z_0;r) \subset A$

$B \subset \mathbb{C}$ is closed iff $\mathbb{C} \setminus B = \{z \in \mathbb{C} \text{ s.t. } z \notin B\}$ is open

Easy to check:
- $\emptyset, \mathbb{C}$ are open
- The union of any collection of open sets is open
- Finite intersection of open sets is open

deMorgan

- $\mathbb{C}, \emptyset$ are closed
- The intersection of any collection of closed sets is closed
- Finite unions of closed sets are closed.

Def $A \subset \mathbb{C}$ is bounded iff $\exists\, N \in \mathbb{R}$ s.t. $|z| \le N\ \forall z \in A$

An open cover of A is a collection of open sets whose union contains A

$K \subset \mathbb{C}$ is compact iff every cover of K by open sets has a finite subcover, i.e. a finite subcollection which also covers K

A set C is not connected (or has a disconnection) iff
$\exists\, \mathcal{U}, V$ open s.t. $(\mathcal{U} \cap C) \cup (V \cap C) = C$
$(\mathcal{U} \cap C) \cap (V \cap C) = \emptyset$

$C \subset \mathbb{C}$ is connected iff it is not (not connected)

Sequences (in $\mathbb{C}$)

Def $\{z_k\} \to L$ iff

Def $\{z_k\}$ Cauchy iff

Thm $\{z_k\}$ Cauchy iff $\{z_k\}$ converges to some limit L

Thm $\{z_k\} \to L$, $\{w_k\} \to M$, $a \in \mathbb{C}$, implies

(i) $\{a z_k\} \to aL$

(ii) $\{z_k + w_k\} \to L + M$

(iii) $\{z_k / w_k\} \to L/M$ provided $w_k \neq 0 \ \forall k$, $M \neq 0$

Sequences and sets

Def: Let B be a set. $z \in \mathbb{C}$ is a limit point of B iff $\exists \{z_k\} \subset B$ s.t. $\{z_k\} \to z$

Thm $B \subset \mathbb{C}$ is closed iff B contains all its limit points

Def $\overline{B} :=$ B union with the limit points of B (It should usually be clear whether we mean closure or conjugate.)

$\overline{B}$ closure

Thm: $\overline{B}$ is closed (and is the intersection of all closed sets containing B)

<u>Thm</u> The following are equivalent for $K \subset \mathbb{C}$

(i) K is compact

(ii) K is closed and bounded

(iii) Every sequence $\{z_k\} \subset K$ has a convergent subsequence $\{z_{k_j}\} \to z_0 \in K$

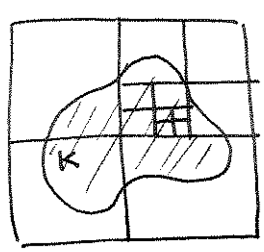
(i) $\Rightarrow$ (ii) : If K is not bounded $\{D(0;n)\}_{n \in \mathbb{N}}$ is a open cover of K without a finite subcover
so K is bounded.

If K is not closed $\exists$ a limit point z_0 of K, not in K.

Then $\{\mathbb{C} \setminus \overline{D(z_0; \frac{1}{n})}\}$ is an open cover of K without a finite subcover

so K is closed $\blacksquare$

(ii) $\Rightarrow$ (iii)



Let $\{z_k\} \subset K$. Since K is bounded, Use diadic subdivision to construct nested squares S_j of side-length $\frac{1}{2^j}$ s.t.

$\exists$ ∞'ly many $\{z_{k_\ell}\} \subset S_j$.

This yields pts $z_{k_j} \in S_j$ $\quad k_1 < k_2 < \dots$

$\{z_{k_j}\}$ Cauchy $\Rightarrow \{z_{k_j}\} \to z_0 \in K$ (since K closed)

(iii) $\Rightarrow$ (i) Uses $\mathbb{C}$ <u>separable</u> (ctble dense subset).

Let $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ cover K. Each $z \in \mathcal{U}_\alpha$ is in some $D(\tilde{z}; \tilde{r}) \subset \mathcal{U}_\alpha$, $\tilde{z} = \tilde{x} + i\tilde{y}$ $\quad \tilde{x}, \tilde{y}, \tilde{r}$ rational.

Thus, $\{D(\tilde{z}; \tilde{r})\}_{D(\tilde{z};\tilde{r}) \subset \mathcal{U}_\alpha \text{ some } \alpha \in \mathcal{A}}$ is an open countable cover of K.

It suffices to prove this ctble cover has finite subcover, since then the corresponding finite # of $\mathcal{U}_\alpha$'s cover K.

rewrite cover as $\{D_j\}_{j \in \mathbb{N}}$. <u>If</u> no finite subcover pick $z_k \notin \bigcup_{j=1}^{k} D_j$

(iii) $\Rightarrow \{z_k\} \to z_0 \in K$

$\Rightarrow z_0 \in D_L$ some $L \Rightarrow \{z_k\}_{k \geq N} \subset D_L$ some N $\divideontimes$ (violate for $k \geq L$). So $\exists$ finite subcover $\blacksquare$