b1.4: 3220 material we need in this course : today: sets & sequences
(and the notation we will use)

**Sets**

\[ D(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \} \]  
"open disk", "\( r \)-neighborhood" \( (\forall r) \)

\[ D(z_0, r) \setminus \{ z_0 \} = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < r \} \]  
"deleted open disk", "deleted \( r \)-neighborhood" \( (\forall r) \)

A \subset \mathbb{C} is a neighborhood of \( z_0 \) iff
\[ \exists r > 0 \text{ s.t. } D(z_0, r) \subset A \]

A \subset \mathbb{C} is open iff \( \forall z_0 \in A \exists r > 0 \text{ s.t. } D(z_0, r) \subset A \)

B \subset \mathbb{C} is closed iff \( C \setminus B = \{ z \in \mathbb{C} \mid z \notin B \} \) is open

Easy to check:
- \( \emptyset, \mathbb{C} \) are open
- The union of any collection of open sets is open
- Finite intersection of open sets is open

\[ \sim \text{DeMorgan} \]

- \( \emptyset, \mathbb{C} \) are closed
- The intersection of any collection of closed sets is closed
- Finite unions of closed sets are closed

**Def** A \( \subset \mathbb{C} \) is bounded iff \( \exists N \in \mathbb{R} \text{ s.t. } |z| \leq N \forall z \in A \)

An open cover of \( A \) is a collection of open sets whose union contains \( A \)

K \( \subset \mathbb{C} \) is compact iff every open cover of \( K \) by open sets has a finite subcover,
\(\text{i.e. a finite subcollection which also covers } K\)

A set \( C \) is not connected (or has a disconnection) iff
\[ \exists U, V \text{ open s.t. } (U \cap C) \cup (V \cap C) = C \]
\[ (U \cap C) \cap (V \cap C) = \emptyset \]

C \( \subset \mathbb{C} \) is connected iff it is not (not connected)
Def \( \{a_n\} \rightarrow L \) iff

Def \( \{a_n\} \) Cauchy iff

Thm \( \{a_n\} \) Cauchy iff \( \{a_n\} \) converges to some limit \( L \)

Thm \( \{a_n\} \rightarrow L, \{w_n\} \rightarrow M, a \in C, \) implies

(i) \( \{a_2n\} \rightarrow aL \)

(ii) \( \{a_n + w_n\} \rightarrow L + M \)

(iii) \( \{a_nw_n\} \rightarrow LM \) provided \( w_n \not= 0 \) \( \forall n, M \not= 0 \)

sequences and sets

Def: Let \( B \) be a set. \( z \in C \) is a limit point of \( B \) iff \( \exists \{a_n\} \subset B \) s.t. \( a_n \rightarrow z \)

Thm: \( B \subset C \) is closed iff \( B \) contains all its limit points

Def: \( \overline{B} := B \) union with the limit points of \( B \) (It should usually be clear whether we mean closure or conjugate.)

Thm: \( \overline{B} \) is closed (and is the intersection of all closed sets containing \( B \))
The following are equivalent for $K \subseteq \mathbb{C}$:

(i) $K$ is compact.

(ii) $K$ is closed and bounded.

(iii) Every sequence $\{z_k\} \subseteq K$ has a convergent subsequence $\{z_{k_j}\} \to z_0 \in K$.

(i) $\Rightarrow$ (ii): If $K$ is not bounded, $\{D(0;n)\}_{n \in \mathbb{N}}$ is an open cover of $K$ without a finite subcover, so $K$ is bounded.

If $K$ is not closed, there exists a limit point $z_0$ of $K$, not in $K$.

Then $\{\mathbb{C} \setminus D(z_0;\frac{1}{n})\}$ is an open cover of $K$ without a finite subcover, so $K$ is closed.

(iii) $\Rightarrow$ (i): Since $K$ is bounded, let $\{S_j\}$ be a nested sequence of squares with side-length $\frac{1}{2^j}$ such that $\exists$ only finitely many $\{z_k\} \subseteq S_j$.

This yields points $z_{k_j} \in S_j, k_1 < k_2 < \ldots$.

$\{z_{k_j}\}$ is Cauchy and $\{z_{k_j}\} \to z_0 \in K$ (since $K$ is closed).

(iii) $\Rightarrow$ (ii): $K$ is separable (a dense subset).

Let $\{U_x\}_{x \in \mathbb{Q}}$ cover $K$. Each $z \in U_x$ is in some $D(z;\frac{1}{n}) \subseteq U_x$.

Thus, $\{D(z;\frac{1}{n})\}_{z \in \mathbb{Q}}$ is a countable cover of $K$.

It suffices to prove this cover has a finite subcover, since then the corresponding finite collection of $U_x$'s cover $K$.

Rewrite the cover as $\{D(z_j;\frac{1}{n})\}_{j \in \mathbb{N}}$. If no finite subcover is found, pick $z_k \notin \bigcup D_{\frac{1}{n}}$.

$\Rightarrow z_k \to z_0 \in K$.

$z_0 \in D_L$ for some $L$ implies $\{z_k\}_{k \geq N} \subseteq D_L$ for some $N$, so $\exists$ finite subcover (violates the $k, L$ assumption).