

Math 4200
Fri 23 Aug

§1.3 "Basic" complex functions

Example 1 $f(z) = az + b$

• polar form

$$z = |z|e^{i\theta} \quad \theta = \arg(z)$$

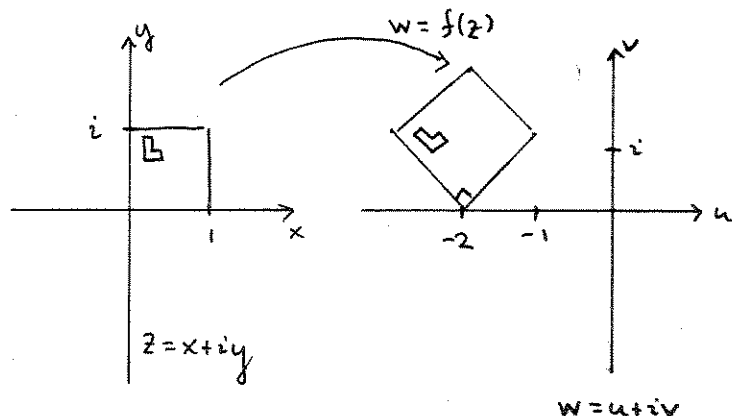
$$a = |a|e^{i\phi} \quad \phi = \arg(a)$$

$$\begin{aligned} f(z) &= |a|e^{i\phi}|z|e^{i\theta} + b \\ &= |a||z|e^{i(\theta+\phi)} + b \end{aligned}$$

f is a composition of

- scaling by a factor of $|a|$
- then, • rotating by an angle ϕ
- then, • translating by b .

e.g. $f(z) = (1+i)z - 2$
 $= \sqrt{2}e^{i\pi/4}z - 2$



The transformation $\Phi: \mathbb{C} \rightarrow \mathbb{R}^2$ is a real vector space isomorphism,
 $\Phi(z) = \begin{bmatrix} \operatorname{Re} z \\ \operatorname{Im} z \end{bmatrix}$ with inverse $\Phi^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + iy$

Each $f: \mathbb{C} \rightarrow \mathbb{C}$ corresponds to $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $F\left(\begin{bmatrix} \operatorname{Re} z \\ \operatorname{Im} z \end{bmatrix}\right) := \begin{bmatrix} \operatorname{Re}(f(z)) \\ \operatorname{Im}(f(z)) \end{bmatrix}$

i.e. if $z = x + iy$ and $f(z) = w = u + iv$
 then $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$

Example 2: Re-analyze Example 1 in terms of the corresponding $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Begin with

$$\begin{aligned} z &= x + iy \\ a &= \alpha_1 + i\alpha_2 \\ b &= \beta_1 + i\beta_2 \end{aligned}$$

$$f(x + iy) = (\alpha_1 + i\alpha_2)(x + iy) + (\beta_1 + i\beta_2)$$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$$

This should look familiar to you!

HW for Fri 31 Aug:

§1.3 1a, 4b, 5a, 6a, 7a, 9, 10, 16b,
21, 23, 30b

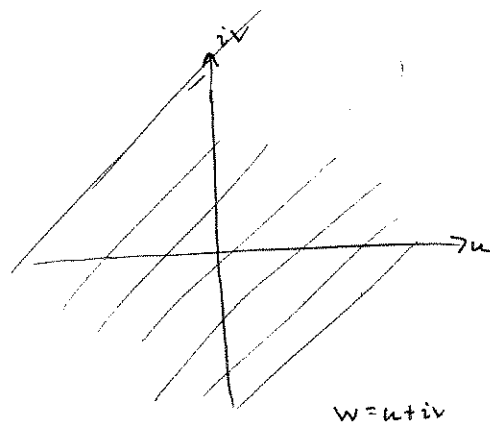
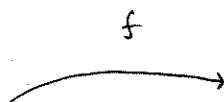
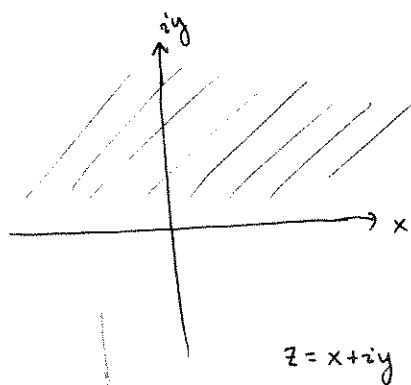
§1.4 1, 2b, 3, 4, 5, 8, 11, 13, 16, 18

(§1.4 is a review of most of the 3220 material we'll need in this course)

Example 3

$$f(z) = z^2$$

$$f(re^{i\theta}) = r^2 e^{2i\theta}$$



$$z^2 = w$$

$$z = \pm \sqrt{w}$$

↑
2 different choices possible.

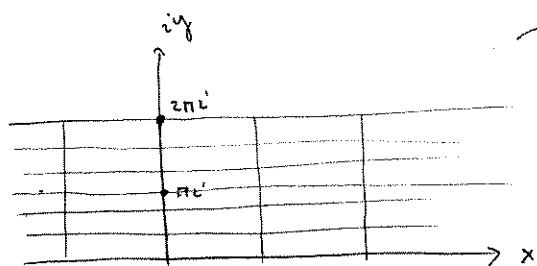
Example 4

$$f(z) = z^3$$

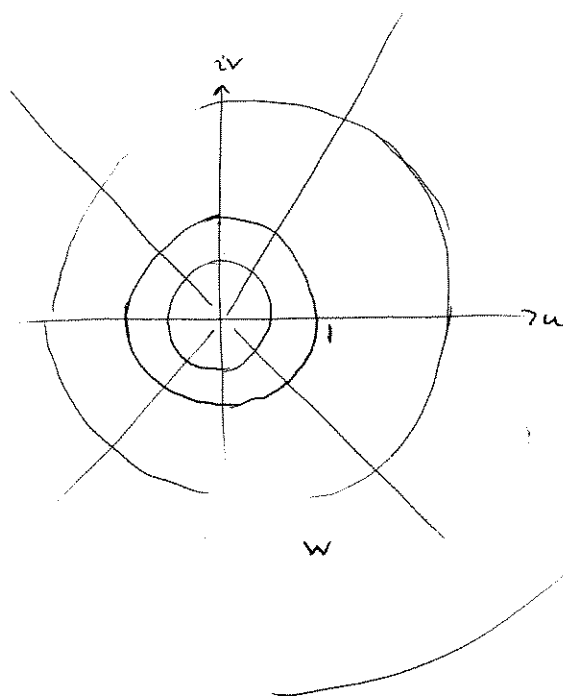
$$f(re^{i\theta}) = r^3 e^{i(3\theta)}$$

Example 5 $z = x + iy$

$$f(z) = e^z := e^x e^{iy}$$



$f \rightarrow$



property: $e^{z+w} = e^z e^w$
did we check this?

inverse fcn?

$$e^z = w$$

$$e^x (\cos y + i \sin y) = u + iv$$

$$\Leftrightarrow \begin{cases} |w| = e^x \Rightarrow x = \log |w| \\ y = \arg(w) \end{cases}$$

so (switching letters),

Example 6

$$\log z := \log |z| + i \arg z$$

if, for example, we restrict $0 \leq \arg z < 2\pi$

then we get a "branch" of the log function which

maps $\mathbb{C} \setminus \{0\}$ 1-1 and onto the (half open) strip above

and $\log(e^z) = z$, $e^{\log z} = z$ with these restrictions

Many other "branch" choices exist!

verify!

also, log property:

$$\log zw = \log z + \log w$$

(up to a multiple of 2π in imag. part)

power functions

For a given branch of $\log$, define

$$z^w := e^{w \log z} \quad (\text{uniquely defines } z^w \text{ for the given branch of } \log)$$

In general z^w is multiple-valued

- If n is integer, $z^n := e^{n \log z}$

$\begin{array}{c} \pm 2\pi k \\ \downarrow \\ n(\log|z| + i \arg z) \end{array}$

$$= e^{n(\log|z| + i \arg z)} = [e^{\log|z| + i(\arg z + 2\pi k)}]^n \quad (e^{z+w} = e^z e^w)$$

$$= z^n \quad \text{since } e^{2\pi i k} = \cos 2\pi k + i \sin 2\pi k = 1 \text{ for } k \text{ an integer}$$

$$\text{i.e. } \underbrace{(z)(z) \dots (z)}_n$$
- If $w = \frac{p}{q}$ is a rational in lowest terms, (p, q no common divisors)

then $z^{p/q}$ is one of the possible q values of (previous) p/q^{th} roots, i.e.

$$z = r e^{i\theta}$$

$$z^{p/q} = r^{p/q} e^{ip(\frac{1}{q}(\theta + 2\pi k))} \quad k = 0, 1, \dots, q-1$$

↖ you can check this on your own!
- If w is irrational,

z^w has ∞ 'ly many possible values, depending on $\log$ branch

(but is uniquely defined after you specify a branch of logarithm)

Trig fncs

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If x is real, $e^{ix} = \cos x + i \sin x$ Eqn 1

$e^{-ix} = \cos x - i \sin x$ Eqn 2

Also

$$\frac{\text{Eqn 1} + \text{Eqn 2}}{2} \Rightarrow \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \quad ; \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{\text{Eqn 1} - \text{Eqn 2}}{2i} \Rightarrow \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad ; \quad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cos z := \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\cosh z := \frac{1}{2}(e^z + e^{-z}) = \cos(iz)$$

$$\sin z := \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\sinh z := \frac{1}{2}(e^z - e^{-z}) = \frac{1}{i} \sin(iz)$$

or

$$\begin{pmatrix} \cos z = \cosh(iz) \\ \sin z = \frac{1}{i} \sinh(iz) \end{pmatrix}.$$

Trig identities hold! (So also trigh identities)

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z+w) = \sin z \cos w + \cos z \sin w$$

$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$

trigh...

$$\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1 \dots$$