Math 4200
Wed Aug 22
JTB 120

Problem session ~~tomorrow~~ Thursday - for HW due Friday
LCB 215, 10:45-11:40

Page 3 Mon. notes (DeMoivre, examples)

Notice that if we define (for θ real)

$$e^{i\theta} := \cos\theta + i\sin\theta \qquad \text{(Euler's formula, motivated by Taylor series)}$$

then the multiplication identity

$$r(\cos\theta + i\sin\theta)\,\rho(\cos\phi + i\sin\phi) = r\rho\,(\cos(\theta+\phi) + i\sin(\theta+\phi))$$

reads prettier:

$$(re^{i\theta})(\rho e^{i\phi}) = r\rho\, e^{i(\theta+\phi)}$$

i.e. the law of exponents holds for $e^{i\theta}$,

$$\boxed{e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}}$$

About solving polynomial equations in $\mathbb{C}$

1. Every complex number $z_0 \neq 0$ has two $\sqrt{\ }$'s, i.e. solutions z to $z^2 = z_0$
(and they are opposites)

<u>proof 1</u>: If $z_0 = re^{i\theta}$
let $z = \sqrt{r}\,e^{i\theta/2}$, $\sqrt{r}\,e^{i(\frac{\theta}{2}+\pi)} = -\sqrt{r}\,e^{i\theta/2}$

<u>proof 2</u>: Write $z_0 = x_0 + iy_0$
$z = x + iy$.

Then $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy \quad (\overset{?}{=} z_0 = x_0 + iy_0)$

level set picture

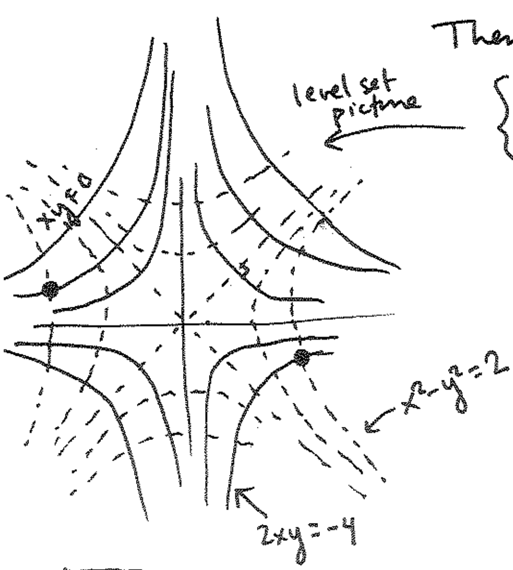
so we want
$$\begin{cases} x^2 - y^2 = x_0 \\ 2xy = y_0 \end{cases}$$

for $x \neq 0$, $y = \dfrac{y_0}{2x}$; subs into 1st eqtn:

$$x^2 - \frac{y_0^2}{4x^2} = x_0 \rightarrow 4x^4 + (4x_0)x^2 - y_0^2 = 0$$

quadratic in x^2, find sol's ...

(also check $y \neq 0$)



$\sqrt{2+4i}$

which proof do you prefer?

2. Every quadratic equation $az^2 + bz + c = 0$ $(a \neq 0)$ $a, b, c \in \mathbb{C}$
has two roots, counting multiplicity

pf with field axioms mimic derivation of quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$, and you know from 1. that you can take $\sqrt{\ }$'s.

3. We'll prove the fundamental theorem of algebra in this course:
(which you've been assuming is true forever, at least for real coef's)

FTA Thm: The poly. $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ $a_i \in \mathbb{C}$
has exactly n roots, counting multiplicity.
(The question of how to find these roots in practice is interesting.)

4. A special polynomial equation is $z^n = 1$
Its solutions are the n^{th} roots of unity. (And there are n of them)

Using Euler, and DeMoivre, write

$$z = |z| e^{i\theta}.$$

solve $|z|^n e^{in\theta} = 1$

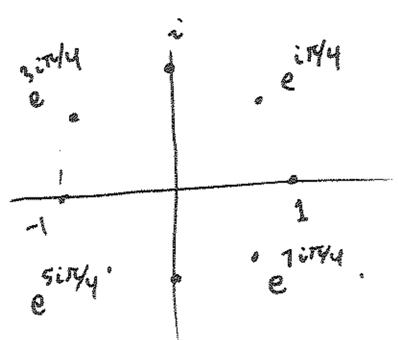
deduce $|z| = 1$

$$e^{in\theta} = 1 = e^{i(2\pi k)} \qquad k \in \mathbb{Z}$$

$$n\theta = 2\pi k \qquad k \in \mathbb{Z}$$

$$\theta = \frac{2\pi k}{n} \ ;$$

$$\theta = 0, \left(\frac{2\pi}{n}\right), \left(\frac{2\pi}{n}\right)2, \left(\frac{2\pi}{n}\right)3, \ldots \frac{2\pi}{n}(n-1)$$



eigth roots of unity.

Some
Identities & Estimates,
for this analysis class, with a complex twist

$$z = x + iy$$
$$\bar{z} = x - iy$$

1) $\overline{z+w} = \bar{z} + \bar{w}$

2) $\overline{zw} = \bar{z}\,\bar{w}$ $\quad \overline{(z/w)} = \bar{z}/\bar{w} \quad (w \neq 0)$

3) $|z|^2 = z\bar{z}$

4) $z = \bar{z}$ iff z real
$z = -\bar{z}$ iff z imag.

5) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$
$\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$

6) $\bar{\bar{z}} = z$

7) $|zw| = |z||w|$
$\left(\left|z/w\right| = \frac{|z|}{|w|} \quad w \neq 0 \right)$

8) $-|z| \leq \operatorname{Re} z \leq |z|$
$-|z| \leq \operatorname{Im} z \leq |z|$
i.e. $|\operatorname{Re} z| \leq |z|$
$|\operatorname{Im} z| \leq |z|$

9) $|\bar{z}| = |z|$

* 10) $|z+w| \leq |z| + |w|$
Δ inequality

* 11) $|z-w| \geq |z| - |w|$
reverse Δ ineq

* 12) $\left| \sum_{i=1}^{n} z_i w_i \right|^2 \leq \left(\sum |z_i|^2 \right)^{1/2} \left(\sum |w_i|^2 \right)^{1/2}$
complex Cauchy-Schwarz