Math 4200
Mon 8/20
JTB 120 11:50–12:40

HW for Friday August 24
1.1 1b, 2c, 4a, 6a, 7, 11 (multiplication properties only), 14, 17a
1.2 1a, 2b, 4, 5, 8, 11, 14, 19

- roll
- organization
- who - background - plans - special

Complex analysis is like Calculus (derivatives and integrals play a major role), except the functions $f(z)$ have complex number domain & range ~ so 1st review the complex plane:

<u>Def</u>: the complex plane $\mathbb{C}$ is defined to be the real vector space $\mathbb{R}^2$ with the additional operation of complex multiplication:

$$\left.\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &:= \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} \\ \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &:= \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \end{aligned} \quad \forall\, x_i, y_i, \alpha \in \mathbb{R} \right\} \text{ real vector space}$$

$$\left.\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + y_1x_2 \end{bmatrix} \right\} \text{ complex multiplication}$$

If we identify $x+iy$ with $\begin{bmatrix} x \\ y \end{bmatrix}$, then the multiplication axiom reads

$$(x_1+iy_1)(x_2+iy_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

which implies in particular that $i^2 = -1$.

$\mathbb{C}$ satisfies the <u>field</u> axioms of Algebra: $\forall z, w, s \in \mathbb{C}$ there holds

$$+ \left\{\begin{aligned} & z+w = w+z \\ & z+(w+s) = (z+w)+s \\ & z+0 = z \\ & z+(-1z) = 0 \end{aligned}\right\} \text{ these are just (some of the) } \mathbb{R}^2 \text{ vector space axioms, checked in 2270}$$

$$\left.\begin{aligned} \cdot \left\{\begin{aligned} &\text{(i)} \quad zw = wz \\ &\text{(ii)} \quad (zw)s = z(ws) \\ &\text{(iii)} \quad 1z = z \\ &\text{(iv)} \quad \exists!\, z^{-1} \text{ s.t. } zz^{-1} = 1 \quad \forall z \neq 0 \end{aligned}\right. \\ \text{distrib. prop} \quad \text{(v)} \quad z(w+s) = zw + zs \end{aligned}\right\} \text{ you check these in HW! :)}$$

Further notation:

If $z = x+iy$ $\quad (x, y \in \mathbb{R})$

$Re(z) := x$ "real part of z"

$Im(z) := y$ "imaginary part of z"

$\bar{z} := x - iy$ "conjugate of z"

$|z| := \sqrt{x^2+y^2}$ "magnitude, or modulus, of z"

Check:

$\overline{zw} = \bar{z}\,\bar{w}$

$|z| = \sqrt{z\bar{z}}$

$\mathbb{C} \quad\quad \mathbb{R}^2$

$x+iy \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$

<u>Geometry of $\mathbb{C}$</u>: Since $\mathbb{C}$ is identified with $\mathbb{R}^2$ we represent it via this identification. Then complex addition corresponds to $\mathbb{R}^2$ vector addition, and multiplication becomes very interesting geometrically, via polar coords.

polar coords:

if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \quad r \geq 0$

then $z := x+iy = r\cos\theta + i r\sin\theta$

$|z| = \sqrt{x^2+y^2} = r$

$\arg(z) := \theta$ (when $r>0$). θ is uniquely determined up to integer multiples of 2π

"argument of z"

<u>Example</u>

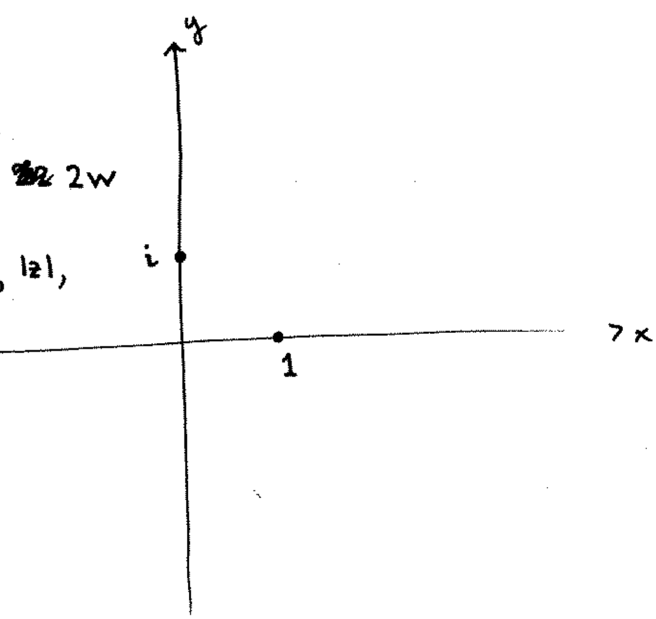
let $z = 2-2i$

$w = \sqrt{3}+i$

(1) Plot $z, w, \bar{z}$, $2w$

$z+w$.

(2) Find $\arg(z)$, $|z|$, and represent them in your picture



multiplication's geometric meaning:

Write $z = r(\cos\theta + i\sin\theta)$ $\quad r = |z|,\ \theta = \arg z$

$w = \rho(\cos\phi + i\sin\phi)$ $\quad \rho = |w|,\ \phi = \arg w$

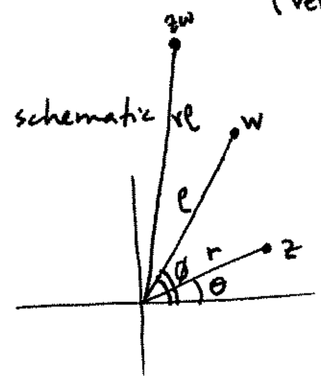
Then $zw = r\rho\,(\cos(\theta+\phi) + i\sin(\theta+\phi))$

So $|zw| = |z||w|$ (could check w/o polar coords)

$\arg(zw) = \arg z + \arg w$

(very neat!)

Check this!!



Corollary: DeMoivre's formula

for $n \in \mathbb{N}$ and $z = r(\cos\theta + i\sin\theta)$

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

pf: by induction

Examples ① Find $\frac{1}{3+4i}$ algebraically & geometrically

② Solve $z^2 = 9i$ alg & geom

③ Find all sol'ns to $z^3 = 27$