Wed 9/7
Math 4200

f $\mathbb{C}$-differentiable at z_0 iff

(1) limit def:

(2) affine approx formula

(3) real-form affine approx. formula (Cauchy-Riemann)

We had checked sum, product, quotient rules for differentiation, were up to the chain rule, which we sketched using 3220 (& you complete in HW)

you! ↓ (quotient)

Chain rule proof 2:

$$(g\circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

pf $f(z_0+h) = f(z_0) + f'(z_0)h + h\tilde{e}(h)$; $\lim_{h\to 0} \tilde{e}(h) = 0$

$w_0 := f(z_0)$

$g(w_0+k) = g(w_0) + g'(w_0)k + k\tilde{\tilde{e}}(k)$; $\lim_{k\to 0} \tilde{\tilde{e}}(k) = 0$

for $k = f'(z_0)h + h\tilde{e}(h)$ get

$$g(f(z_0+h)) = g(f(z_0)) + g'(f(z_0))\left[f'(z_0)h + h\tilde{e}(h)\right] + \left[hf'(z_0) + h\tilde{e}(h)\right]\tilde{\tilde{e}}(k)$$

$$= g(f(z_0)) + \underbrace{g'(f(z_0))f'(z_0)}\,h \quad + h\underbrace{\left[g'(f(z_0))\tilde{e}(h) + (f'(z_0)+\tilde{e}(h))\tilde{\tilde{e}}(k)\right]}_{\tilde{E}(h)}$$

$$= g(f(z_0)) + Ah + h\tilde{E}(h)$$

$\therefore$ $g\circ f$ diffble with deriv. $A = g'(f(z_0))f'(z_0)$ ∎

$\lim_{h\to 0} \tilde{E}(h) = g'(f(z_0))\cdot 0 + f'(z_0)\cdot 0 = 0$

as $h\to 0$, $k\to 0$, so $\tilde{\tilde{e}}(k)\to 0$.

example $\frac{d}{dz} e^{\cos(10z)} =$

? First check e^z is complex differentiable... easiest to use real form!! → see page 4

? Do you know $\frac{d}{dz}\cos z$?

<u>Chain rule for curves</u>: If $\gamma:(a,b)\to\mathbb{C}$ is diff'ble $\left(\begin{array}{l}\gamma(t):=x(t)+iy(t)\\ \gamma'(t):=x'(t)+iy'(t)\end{array}\right)$
and if f is analytic at $\gamma(t)$ then

$$\frac{d}{dt}f(\gamma(t))=f'(\gamma(t))\,\gamma'(t)$$

<u>proof</u>: use affine approx formula $h=\gamma(t+\Delta t)-\gamma(t)$:

$$f(\gamma(t+\Delta t))=f(\gamma(t))+f'(\gamma(t))\left[\gamma(t+\Delta t)-\gamma(t)\right]+h\,\tilde{e}(h)$$

$$\underbrace{\frac{f(\gamma(t+\Delta t))-f(\gamma(t))}{\Delta t}}_{\downarrow\ \frac{d}{dt}f(\gamma(t))}=f'(\gamma(t))\underbrace{\left[\frac{\gamma(t+\Delta t)-\gamma(t)}{\Delta t}\right]}_{\downarrow\ \gamma'(t)}+\underbrace{\frac{h}{\Delta t}}_{\downarrow\ \gamma'(t)}\underbrace{\tilde{e}(h)}_{\downarrow\ 0}$$

■

<u>interpretation</u>: consider curves $\gamma(t)$ passing thru z_0, and
their images $f(\gamma(t))$ passing thru $f(z_0)$.
The corresponding tangent vectors are all rotated and dilated by the same factors $(|f'(z_0)|, \arg f'(z_0))$.
In particular if

$$\alpha(0)=\gamma(0)=z_0$$

then $\measuredangle\, \alpha'(0),\gamma'(0)=\measuredangle\, (f\circ\alpha)'(0),(f\circ\gamma)'(0)$

such maps are called <u>conformal</u> when the scaling factor $\neq 0$ $(f'(z_0)\neq 0)$

(this is another way of thinking of the L-box pictures)

examples: CR revisited:

$$\frac{\partial f}{\partial x}=f'\cdot 1$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}f(x+iy)=f'i$$

so $\frac{\partial f}{\partial y}=i\frac{\partial f}{\partial x}$; $u_y+iv_y=i(u_x+iv_x)$

$$\boxed{\begin{array}{l}u_x=v_y\\ u_y=-v_x\end{array}}$$ rectangular CR

or in polar coords:

$$\frac{\partial f}{\partial r}=\frac{\partial}{\partial r}f(re^{i\theta})=f'e^{i\theta}$$

$$\frac{\partial f}{\partial \theta}=\frac{\partial}{\partial \theta}f(re^{i\theta})=f'rie^{i\theta}$$

so $\frac{\partial f}{\partial \theta}=ri\frac{\partial f}{\partial r}$; $u_\theta+iv_\theta=ri(u_r+iv_r)$

$$\boxed{\begin{array}{l}u_\theta=-rv_r\\ ru_r=v_\theta\end{array}}$$ polar CR

<u>example</u>: $f(z)=z^2=(x^2-y^2)+i(2xy)$
$=r^2e^{2i\theta}=r^2\cos 2\theta+i(r^2\sin 2\theta)$

Real analysis facts (Ma 3220)

Def a neighborhood A of $\vec{x}_0 \in \mathbb{R}^n$ is any open set containing $\vec{x}_0$

Def $F: \underset{\mathbb{R}^n}{\overset{\cup}{A}} \to \mathbb{R}^m$ is differentiable at $x_0 \in A$ iff for $\|h\|$ sufficiently small

$$F(x_0 + h) = F(x_0) + [Df(x_0)]h + E(h)$$

where $\displaystyle\lim_{h \to 0} \frac{\|E(h)\|}{\|h\|} = 0$

and where $[Df(x_0)]_{m \times n}$ is the matrix $\left[\frac{\partial F_i}{\partial x_j}(x_0)\right]$.

Theorem (Real variables chain rule) (repeat!)

If
$$F: A \to \mathbb{R}^m, \quad A \subset \mathbb{R}^n$$
$$G: B \to \mathbb{R}^k, \quad B \subset \mathbb{R}^m$$

if $x_0 \in A$, F diffble at x_0
& $F(x_0) \in B$, G diffble at $F(x_0)$

then $G \circ F$ is diffble at x_0 and

$$[D(G \circ F)(x_0)]_{k \times n} = [DG(F(x_0))]_{k \times m} [DF(x_0)]_{m \times n}$$

Theorem If F and all partial derivatives $\frac{\partial F_i}{\partial x_j}$ are continuous in a nbhd of $\vec{x}_0$ then F is differentiable at $\vec{x}_0$. In this case we call F "C^1-differentiable" or just C^1.

Theorem (Inverse function theorem).

If $F: A \to \mathbb{R}^n$, $A \subset \mathbb{R}^n$, is C^1 differentiable in a nbhd of $x_0 \in A$ and if $[Df(x_0)]_{n \times n}$ is non-singular, then

$\exists$ open sets $\mathcal{U}, V$ s.t
$x_0 \in \mathcal{U}$, $F(x_0) \in V$
$F: \mathcal{U} \to V$ bijective (1-1 & onto)
$F^{-1}: V \to \mathcal{U}$ is C^1 differentiable

And $[DF^{-1}(F(x))] = [DF(x)]^{-1}$

Applications to $\mathbb{C}$-analysis:

<u>Theorem</u>: If $F\binom{x}{y} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is continuous and has continuous partial derivs u_x, u_y, v_x, v_y in a nbhd of $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

and if C-R eqtns hold at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$:

$$u_x = v_y$$
$$u_y = -v_x$$

Then $f(x+iy) := u(x,y) + i v(x,y)$

is $\mathbb{C}$ analytic at $x_0 + i y_0 = z_0$

<u>pf</u>: F is C^1 in a nbhd of $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ so is diffble (affine approx holds)

Cauchy-Riemann eqtns hold at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ so $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ at $\binom{x_0}{y_0}$

therefore the complex affine approx holds for $f = u + iv$. (at $x_0 + iy_0$)

Therefore $f'(z_0)$ exists (and equals $a+bi$
$= u_x + i v_x$
$= v_y - i u_y$).

<u>example</u>: In your HW you verify that
$e^z = e^x(\cos y + i \sin y)$ is analytic. Shall we try it?

<u>Theorem</u>: (Inverse function theorem).

If f is $\mathbb{C}$-analytic in a nbhd of z_0, with continuous partial derivatives u_x, u_y, v_x, v_y and if $f'(z_0) \neq 0$ then

$\exists$ open sets $\mathcal{U}, V$ s.t.

$z_0 \in \mathcal{U}$, $f(z_0) \in V$

$f: \mathcal{U} \to V$ a bijection

$f^{-1}: V \to \mathcal{U}$ $\mathbb{C}$-analytic.

<u>pf</u>: Apply the real variables IVT to $F\binom{x}{y} : \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

(note $\left[DF\binom{x_0}{y_0}\right] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $f'(z_0) = a+bi$
so $f'(z_0) \neq 0 \Rightarrow \det > 0$.)

Then note that

$$F^{-1} \circ F \binom{x}{y} = \binom{x}{y}$$

$\Rightarrow$ $DF^{-1} \circ DF = id$ (chain rule)

$\Rightarrow$ $DF^{-1} = [DF]^{-1}$ is also a rot-dil matrix, so CR hold for F^{-1}, then apply thm at page

Harmonic conjugates

Let $f = u + iv$ be C^2 (partials thru 2nd order are cont) in a nbhd of z_0

CR: $u_x = v_y$, $u_y = -v_x$ $\Rightarrow$ $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$ $\quad v \in C^2 \Rightarrow v_{yx} = v_{xy}$

so $u_{xx} + u_{yy} = 0$ — this is def. of u is harmonic.

"Δu" $= 0$ — Laplace eqtn.

(v is also harmonic).

so,

f analytic, $f \in C^2 \Rightarrow \operatorname{Re} f, \operatorname{Im} f$ are harmonic.

There is a local converse:

Thm If u is C^2 in a nbhd of z_0 and is harmonic.
Then (in a nbhd of z_0) $\exists!$ harmonic fcn $v(x,y)$ satisfying
up to additive const

$$* \begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$$

and then
$f(x+iy) := u + iv$ is $\mathbb{C}$-analytic in a nbhd. of z_0

pf the integrability condition for $*$ (ref Green's thm, conservative vector fields Math 2210)
is that $(v_x)_y = (v_y)_x$! $-u_{yy} = u_{xx}$ ✓

Once you find v,
$u + iv$ is C^1 & sats CR $\Rightarrow$ analytic.

example
Let $u(x,y) = xy$
Find v, identify f: