Math 4200
Wed 28 Sept

§2.4, 1st half:

Index thm, Cauchy thm

↑ "winding number"

↑ THE most important basis theorem in complex analysis

Notes: postpone §2.4 HW to next assignment
Exam Monday will cover 1–2.3 only.
So, in tomorrow's problem session, after we deal with remaining §2.3 HW feel free to ask more general questions.

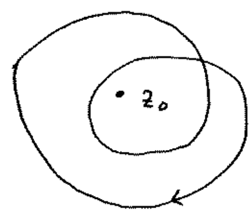
- technical point(s) page 6 Monday (discuss these first)

- Index (or a winding number) of a closed path γ (piecewise C^1) wrt z_0 ($z_0 \notin$ range γ): how many times does γ wind around z_0 in counterclockwise direction?

$I(\gamma; z_0)$

$I(\gamma; z_0) = 1$

$I(\gamma; z_0) = 0$



$I(\gamma; z_0) = -2$

Def: $I(\gamma; z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz$

- $I(\gamma; z_0)$ is a homotopy invariant in $\mathbb{C} \setminus \{z_0\}$ wrt closed curves, by the deformation theorem (Theorem 3 Wed.).

- If $\gamma(t) = z_0 + e^{i(2\pi n t)}$ $0 \le t \, 1$, $n \in \mathbb{Z}$

then $\gamma'(t) = 2\pi n i \, e^{i(2\pi n t)}$

$z - z_0 = e^{i(2\pi n t)}$

so, $\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz = \frac{1}{2\pi i} \int_0^1 2\pi n i \, dt = n$ ✓

Theorem $I(\gamma; z_0)$ is always an integer $n \in \mathbb{Z}$

In fact $I(\gamma; z_0) = n$ if and only if $\gamma(t)$ is homotopic in $\mathbb{C} \setminus \{z_0\}$ to $\gamma^n(t) = z_0 + e^{i(2\pi n t)}$ $\quad 0 \le t \le 1$ (as closed curves).

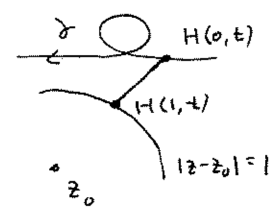
proof: (see text for more elegant but obscure version of part of this theorem).

Step 1 $\gamma(t)$ is homotopic in $\mathbb{C} \setminus \{z_0\}$ to a piecewise smooth curve $\gamma_1(t)$ lying on the unit circle $|z - z_0| = 1$

pf. $H(s,t) = (1-s)\gamma(t) + s\left[\dfrac{\gamma(t) - z_0}{|\gamma(t) - z_0|}\right]$

$s = 0: \ \gamma(t)$

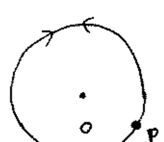
$s = 1: \ \dfrac{\gamma(t) - z_0}{|\gamma(t) - z_0|}$



Step 2: So $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{1}{z - z_0}\,dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z - z_0}\,dz$

Step 3: Let $\gamma_2(t) = \gamma_1(t) - z_0$ (translate by z_0). Then $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_1} \frac{1}{z - z_0}\,dz = \frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{z}\,dz$

(translates discussion back to origin).



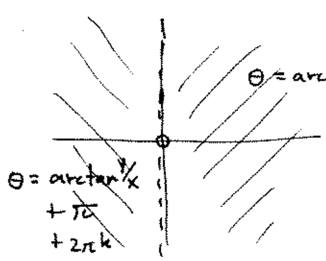
γ_2 is going back & forth around the unit circle.

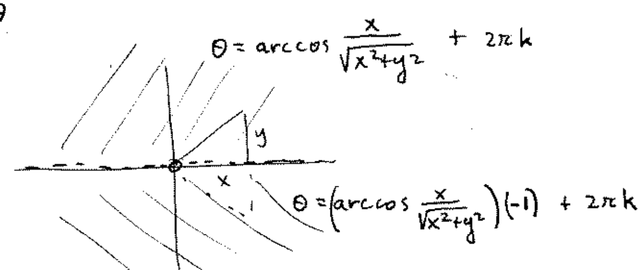
If we can $\gamma_2(t)$ is homotopic to some $\gamma^n(t) = e^{i(2\pi n t)}$ then thm follows.

To do this we must figure out how often γ_2 winds around!

Use (and piece together) local antiderivatives $\log z$ to $\frac{1}{z}$!

$\log z = \log|z| + i\underbrace{\arg z}_{\Theta}$

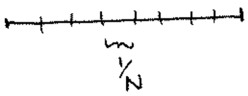


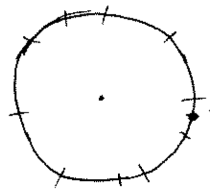


$\arg z = \Theta$ has smooth (C^∞) definitions on each half plane, up to $2\pi k$, and these agree on quadrant overlaps.

[redacted]

<u>step 4</u> pick N s.t. $\gamma_2\left(\left[\frac{k-1}{N},\frac{k}{N}\right]\right)$ lies in <u>one</u> of the 4 coord half s $\forall k = 1,\dots N$
(by uniform continuity





γ_2 (going back & forth on $|z|=1$)

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{1}{z}\,dz = \sum_{k=1}^{N}\frac{1}{2\pi i}\int_{\tilde{\gamma}_k}\frac{1}{z}\,dz = \sum_{k=1}^{N}\frac{1}{2\pi i}\left(\log|z| + i\arg z\right)\Big]_{\gamma(\frac{k-1}{N})}^{\gamma(\frac{k}{N})}$$

$\tilde{\gamma}_k \downarrow \gamma_2\big|_{[\frac{k-1}{N},\frac{k}{N}]}$

key point: choose $\arg z$ continuously, hence smoothly along all of γ_2 starting on the subinterval $[0, 1/N]$, using the smooth "patches" for half planes on page 2, and matching $\arg z = \theta$ at right & left endpoints. You are actually defining a function $\theta(\underline{t})$.

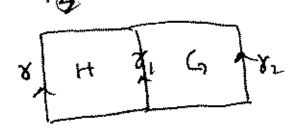
thus the series on the right telescopes, and we get

$$\frac{1}{2\pi i}\left(\cancel{\log|\gamma(1)| - \log|\gamma(0)|} + i\left(\arg(\gamma(1)) - \arg(\gamma(0))\right)\right)$$

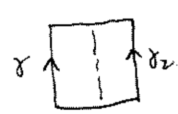
the choice of arg along γ, actually along $0 \le t \le 1$ and is some multiple $2\pi n$ of 2π

$= n$, some $n \in \mathbb{Z}$.

Thus $I(\gamma; z_0) = n$, some $n \in \mathbb{Z}$.
Further, $\gamma_2(t)$ is homotopic to $e^{2\pi i n t}$, so γ is homotopic to $z_0 + e^{2\pi i n t}$
("homotopic" to is transitive)



$$H(s,t) = e^{i\left[(1-s)\theta(t) + s(2\pi n t)\right]}$$



$$H(0,t) = e^{i\theta(t)} = \gamma_2(t)$$
$$H(1,t) = e^{i(2\pi n t)}$$

∎

Now we can prove one of the most important results of complex analysis!

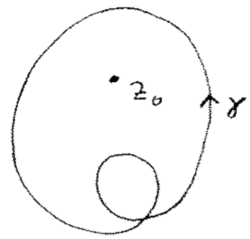
Cauchy Integral formula

Let f be analytic in A, γ a closed curve in A, homotopic to a point (homotopy & pt in A)

Let $z_0 \notin \gamma(I)$

Then

$$\boxed{f(z_0) \cdot I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz}$$



the values of an analytic fcn "inside" a curve are completely (& almost explicitly) determined by the values on γ

[if you've had PDE's this should remind you of the Dirichlet problem for harmonic fcns].

proof:

Let $g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$

g is diff'ble in $A \sim \{z_0\}$ and continuous in all of A.

Suppose the deformation thm held for g (old hypothesis: g diff'ble on A)

Then

$$\int_\gamma g(z)\, dz = \int_{\text{point}} g(z)\, dz = 0$$

$$\| $$

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad \text{(since } \gamma \text{ avoids } z_0\text{)}$$

$$\| $$

$$0 = \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) \underbrace{\int_\gamma \frac{1}{z - z_0}\, dz}_{2\pi i\, I(\gamma; z_0)}$$

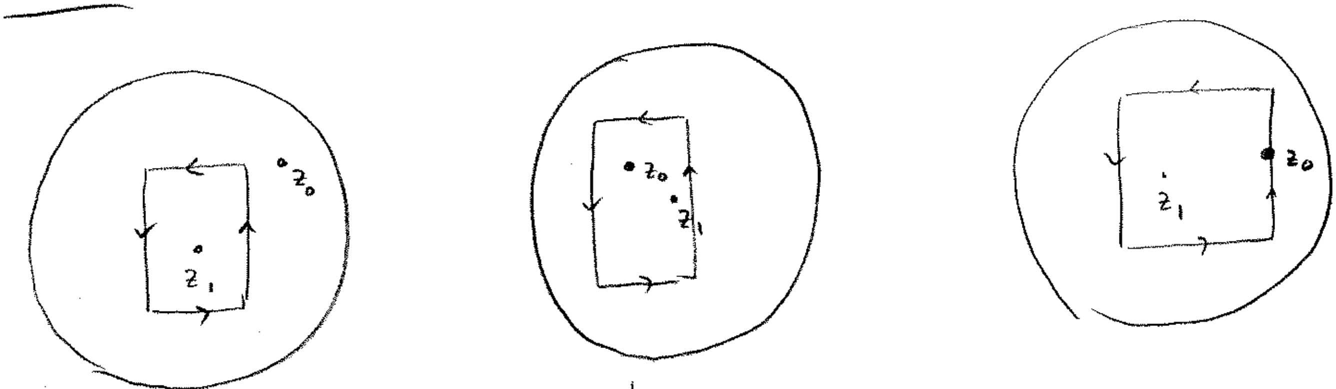
What about the fact that g is only known to be cont at z_0?

Well,

$$\text{deformation thm} = \underbrace{\text{local antideriv thm}}_{\substack{\| \\ g \text{ cont } + \text{ rectangle lemma}}} + \text{analysis}$$

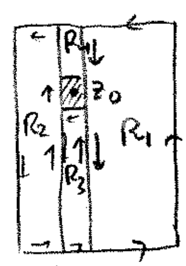
Can we still prove the rectangle lemma if g analytic in $D(z_1, r)$ *except* at one point $z_0 \in D$?

3 cases:

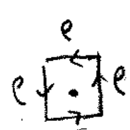


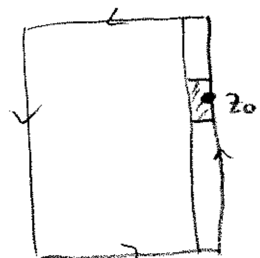
old proof still works!

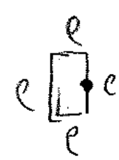
$$\int_{\partial R} g(z)\,dz = 0.$$



$$\int_{\partial R} g(z)\,dz = \int_{\partial R_e} g(z)\,dz$$







same argument.

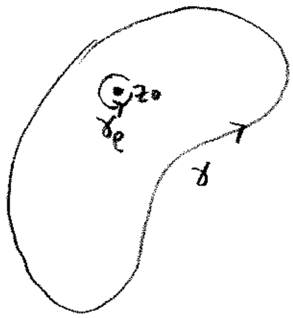
$$\left| \int_{\partial R_e} g(z)\,dz \right| \le \int_{\partial R_e} |g(z)|\,|dz|$$

$$\le \left(\max_{\partial R_e} |g(z)|\right) 4e$$

$\to 0$ since g is bounded near z_0 (since cont. there)

(Would be O.K. even if g blew up at a "sublinear" rate)

In case γ is the boundary of a simply connected domain A, $z_0 \in A$ (so) & $I(\gamma; z_0) = 1$, there is a more intuitive proof of Cauchy Integral formula:



$\gamma_\rho = \{|z-z_0|=\rho\}$ ρ small s.t. $\gamma_\rho \subset A$

Deformation thm

$$\Rightarrow \oint_\gamma \frac{f(z)}{z-z_0}\,dz = \oint_{\gamma_\rho} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_{\gamma_\rho} \frac{f(z_0)}{z-z_0}\,dz + \underbrace{\int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz}$$

$$\int_{\gamma_\rho} \frac{f(z_0)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

(our favorite computation!)

$$\left|\quad\right| \le \int_{\gamma_\rho} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| |dz|$$

$$\le \left(\max_{\gamma_\rho} |f(z)-f(z_0)|\right) \frac{1}{\cancel{\rho}}\, 2\pi \cancel{\rho}$$

$$\downarrow$$

$$0$$

as $\rho \to 0$

$$\Rightarrow \int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$