Monday 9/26 : Finish §2.3

On Friday we proved the local antidifferentiation result:

<u>Theorem 1</u>: If $f: D(z_0, r) \to \mathbb{C}$ is complex differentiable $\left(\forall z \in D(z_0, r)\right)$
then $\exists$ antiderivative $F(z)$ $\quad (F' = f \;\; \forall z \in D)$.

This followed from the rectangle lemma

<u>lemma</u> $\forall \; R = [a,b] \times [c,d] \subset D(z_0, r)$
$\gamma = \partial R$

Then $\displaystyle\oint_\gamma f(z)\,dz = 0$

We used the differentiability of f to deduce the rectangle lemma,
<u>but</u> once we knew the rectangle lemma, we only needed the <u>continuity</u> of f to construct F. (This will be important in §2.4)

Today we will use Theorem 1 to prove

defined precisely on page 3 today

<u>Theorem 2</u> (Antiderivative Theorem) Let A be open and <u>simply connected</u>.
Let $f: A \to \mathbb{C}$ be complex differentiable $(\forall z \in A)$.
Then $\exists$ antiderivative $F: A \to \mathbb{C}$, $F' = f$.

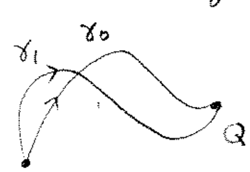
Cor

<u>Theorem 3</u> (Deformation Theorem) Let A be an open set (<u>not</u> necessarily simply connected)
Let $f: A \to \mathbb{C}$ complex differentiable.
Let γ_0, γ_1 piecewise C' curves which are <u>homotopic in A</u>

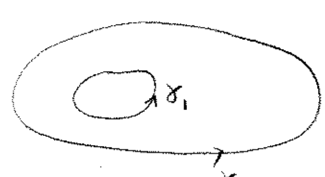
see page 2

(a) If they are homotopic with fixed endpoints
<u>or</u>
(b) if they are homotopic as closed curves,
<u>then</u>

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$



fixed endpoint homotopic curves



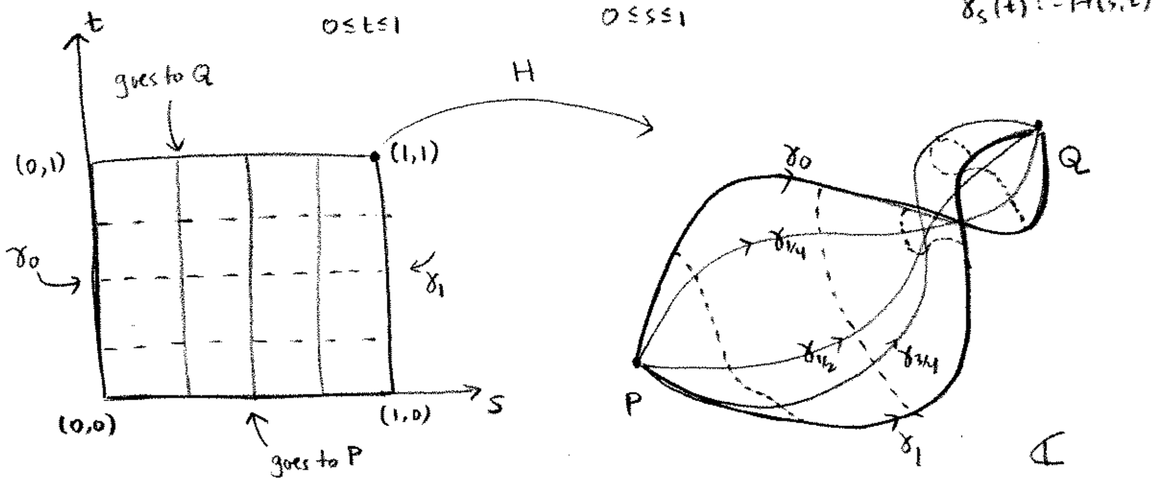
curves homotopic as closed curves

<u>Def</u>: Let $\gamma_0 : [0,1] \to A \subset \mathbb{C}$, $\gamma_0(0) = P$, $\gamma_0(1) = Q$
$\gamma_1 : [0,1] \to A \subset \mathbb{C}$ $\quad \gamma_1(0) = P \quad \gamma_1(1) = Q$

Then γ_0, γ_1 are <u>homotopic in A</u>, with <u>fixed endpoints</u>, if $\exists$ a continuous $H : [0,1] \times [0,1] \to A$ (called the <u>homotopy</u>) s.t.

$$H(0,t) = \gamma_0(t) \qquad H(s,0) = P$$
$$H(1,t) = \gamma_1(t) \qquad H(s,1) = Q$$
$$0 \le t \le 1 \qquad 0 \le s \le 1$$

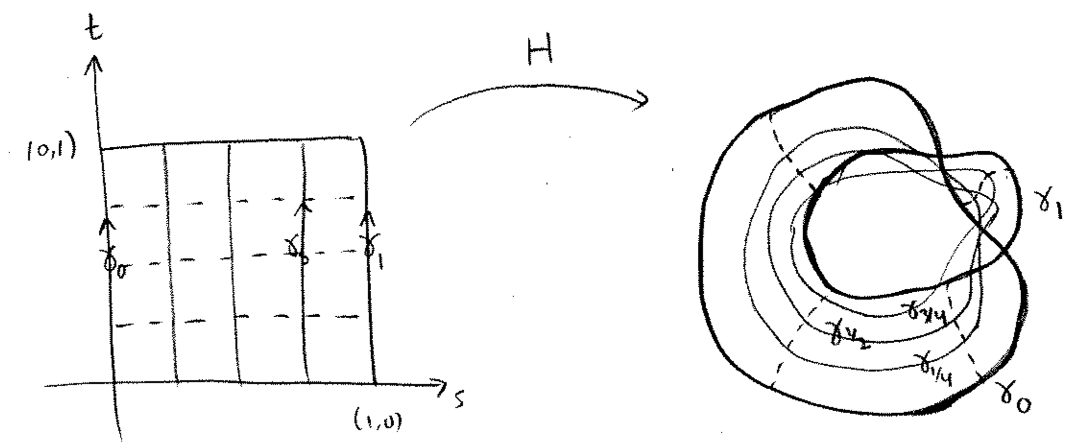
family of curves
$\gamma_s(t) := H(s,t)$



<u>Def</u> Let γ_0, γ_1 as above, except that they are closed curves, $\gamma_0(0) = \gamma_0(1) = P$
$\gamma_1(0) = \gamma_1(1) = Q$

Then γ_0, γ_1 are homotopic in A as closed curves if $\exists$ continuous homotopy $H : [0,1] \times [0,1] \to A$ s.t.

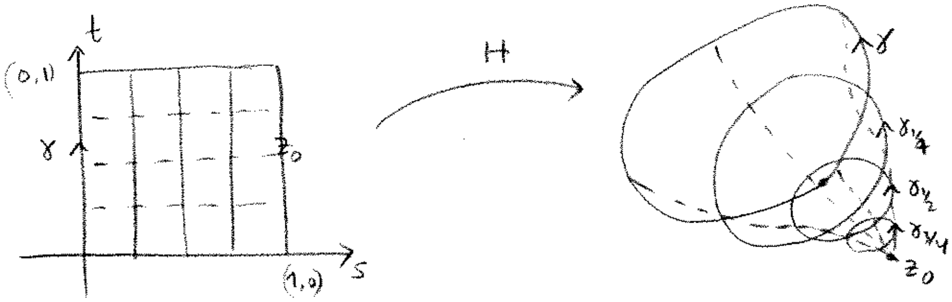
$$H(s,0) = H(s,1) \quad , \text{ i.e. each } \gamma_s(t) = H(s,t) \text{ is closed.}$$
$$0 \le s \le 1$$



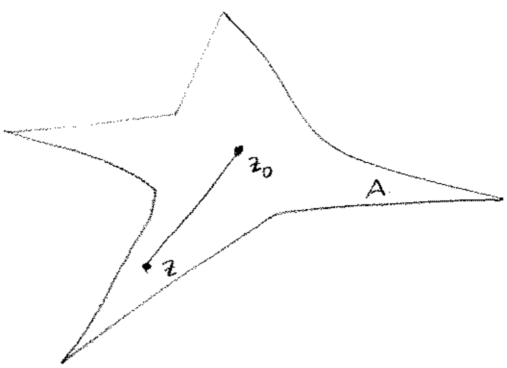
Def A connected open set A is <u>simply connected</u> iff every closed curve $\gamma : [0,1] \to A$ is homotopic as a closed curve to some point $z_0 \in A$, i.e.

$\exists\ H : [0,1] \times [0,1] \to A$ continuous

$$H(0,t) = \gamma(t) \qquad H(s,0) = H(s,1)$$
$$H(1,t) = z_0 \qquad 0 \le s \le 1$$
$$0 \le t \le 1$$



<u>Example</u> A is called <u>starshaped</u> iff $\exists\ z_0 \in A$ s.t. $\forall z \in A$, $0 \le s \le 1$
$(1-s)z + sz_0 \in A$

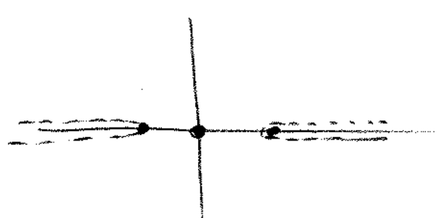


If A is starshaped and $\gamma : [0,1] \to A$ define

$$H(s,t) = (1-s)\gamma(t) + sz_0 \, ;$$

shrinks γ to z_0 as $0 \le s \le 1$.
So starshaped domains are simply connected.

<u>example</u> domain of our old friend $\sqrt{z^2-1}$ is starshaped wrt 0.



[illegible] can prove our theorems!

Theorem 2 (Antidifferentiation) follows from Theorem 3b as follows:

Let A be simply connected, $f: A \to \mathbb{C}$ analytic. We will show path independence, which we know implies $\exists F$.

Let $\gamma_1 : [a,b] \to A \qquad \gamma_1(a) = \gamma_2(c) = P$
$\quad\;\; \gamma_2 : [c,d] \to A \qquad \gamma_1(b) = \gamma_2(d) = Q$

Let $\gamma = \gamma_1 - \gamma_2$, suitably reparameterized so $\gamma : [0,1] \to A$

Then γ is closed since $\gamma(0) = \gamma(1) = P$

Since A is simply connected γ is homotopic to z_0 in A (as closed curves)

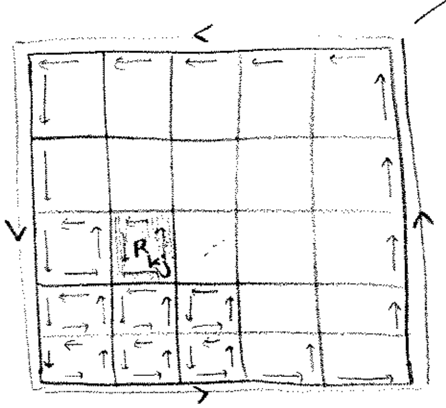
Thm 3b $\Rightarrow \displaystyle\int_\gamma f(z)\,dz = \int_{z_0} f(z)\,dz = 0$ $\quad (dz = 0!)$

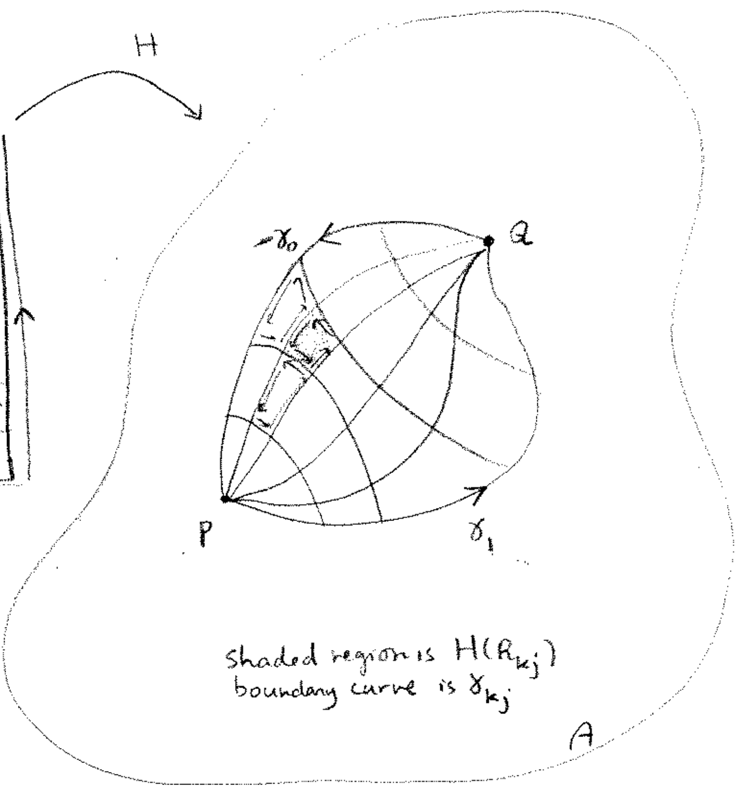
$$\Rightarrow \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = 0 \quad \blacksquare$$

notice, we only need contour integrals over closed curves $= 0$ to deduce full path ind (& antideriv)

Now for the deformation thm!

Here's the picture - let's fill in the words...





$R = [0,1] \times [0,1]$

$\gamma = H(\partial R)$, oriented counterclockwise

subrectangles R_{kj}

$\gamma_{kj} = H(\partial R_{kj})$ (c.c. orient)

picture for fixed pt homotopy

Claim $\displaystyle\int_\gamma f(z)\,dz = 0$, for both fixed pt & closed curve homotopies

Justify the claim:

(1) $\displaystyle\int_\gamma f(z)\,dz = \sum_{k,j} \int_{\gamma_{kj}} f(z)\,dz$ (interior cancellation)

We will show that for a fine enough partition of $[0,1]\times[0,1]$ each $\displaystyle\int_{\gamma_{kj}} f(z)\,dz = 0$, using Theorem 1 (local antiderivative), and analysis:

(2) $H([0,1]\times[0,1])$ is compact subset of A so

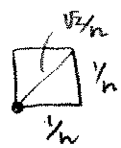
$$\exists\ \varepsilon>0 \text{ s.t. } D(z,\varepsilon)\subset A \quad \forall z\in H([0,1]\times[0,1])$$

pf: (3220, see page 52 of text)

(3) H is continuous on a compact set so it is uniformly continuous

Thus for ε in (2) $\exists\ \delta$ s.t. $\|(s,t)-(s',t')\|<\delta \Rightarrow |H(s,t)-H(s',t')|<\varepsilon$

(4) partition $[0,1]\times[0,1]$ into n^2 squares, $\Delta s=\Delta t=\frac{1}{n}$.

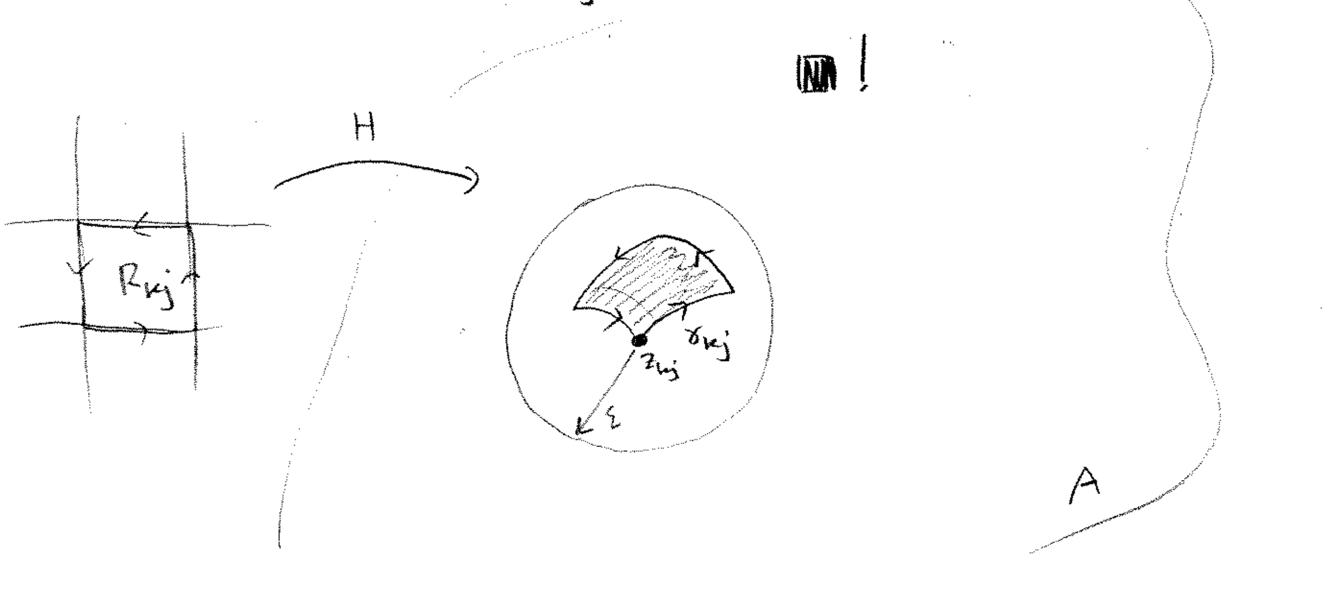


pick n large s.t. $\frac{\sqrt{2}}{n}<\delta$ in (3)

$$\Rightarrow \text{each } H(R_{kj})\subset D(z_{kj},\varepsilon)\,, \quad z_{kj}=H(s_k,t_j)$$

(5) By local antiderivative thm in $D(z_{kj},\varepsilon)$,

$$\int_{\gamma_{kj}} f(z)\,dz = 0$$



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$$\int_\gamma f(z)\,dz = 0$$

if H was a fixed pt homotopy deduce $\displaystyle\int_{\text{"}P\text{"} + \gamma_1 + \text{"}Q\text{"} - \gamma_0} f(z) = 0 \Rightarrow \int_{\gamma_1} f(z)\,dz - \int_{\gamma_0} f(z)\,dz = 0$

<u>Thm 3a</u>

if H was a closed curve homotopy,

deduce $\displaystyle\int_{\text{"}H(s,0)\text{"} + \gamma_1 - \text{"}H(s,1)\text{"} - \gamma_0} f(z)\,dz = 0$

but $H(s,0) = H(s,1)$!

$$\Rightarrow \int_{\gamma_1} f(z)\,dz - \int_{\gamma_0} f(z)\,dz = 0$$

<u>Thm 3b</u>

∎

"small" technical pt: homotopy was only continuous, not C^1.
(But can fix this with a polygonal mesh thru the image vertices)