Where we are:

- If \( A \) is any \underline{open} set and if \( f(z) \) has an antiderivative \( F(z) \) in \( A \) (continuous), then contour integrals \( \int_C f(z) \, dz \) in \( A \) are path independent.

- Path independence in a \underline{connected} open set is equivalent to \( \exists F \text{ s.t. } F'(z) = f(z) \; \forall z \in A \) for \( \int_C f(z) \, dz \)

- If \( A \) is open and \underline{simply connected} and \( f \in C^1(A) \) is analytic, then integrals \( \int_C f(z) \, dz \) are path independent, so also \( \int_C f(z) \, dz \) (used Green's Thm)

We have proven the \textbf{1st} two theorems carefully, but the third one only \underline{semi} carefully since we didn't precisely understand simply connected, and couldn't show path independence for all paths.

We've proven the first two theorems carefully, but the third one only semi-carefully. Since we didn't precisely understand simply connected, and couldn't show path independence for all paths.

The goal of \( \text{Sec. 2.3} \) is to understand this theorem and the \textbf{deformation theorem} precisely (about switching contours w/o changing the value of the contour integral).

Key Step:

Theorem:

Local

Antiderivative (is special case) Let \( D(\mathbf{z}_0, r) = \{ z \mid |z - z_0| < r \} \). Let \( f : D(\mathbf{z}_0, r) \to \mathbb{C} \) be complex differentiable \( \forall z \in D(\mathbf{z}_0, r) \). Then \( \exists F : D \to \mathbb{C} \) s.t. \( F'(z) = f(z) \; \forall z \in D \).

Note: do not need \( f \in C^1 \) or Green's Thm.

The key step Theorem will follow from:

Lemma: Let \( R = [a, b] \times [c, d] \) be any rectangle (sides // to coord dirs) in \( D \) Let \( \gamma = \partial R \) (counterclockwise orientation)

Then \( \oint_{\gamma} f(z) \, dz = 0 \)
Assuming lemma for now, define local antidiff them:

Define \( F(w) = \int f(z) \, dz \) where \( \gamma_w = \gamma_{w_1} + \gamma_{w_2} \)

\[ \begin{align*}
\text{with} & \quad \gamma_{w_1} + \gamma_{w_2} \\
& \quad \text{horiz} \quad \text{vert}
\end{align*} \]

Must compare \( F(w+h) \) to \( F(w) \).

Write \( \gamma_w = \gamma_1 + \gamma_2 \).

\( \gamma_{w+h} = \alpha_1 + \alpha_2 + \beta_2 \)

\( \text{with} \quad \alpha_1 + \beta_1 = \gamma_1 \)

and \( \beta_3 + \beta_2 \) the horizontal then vertical displacement curve from \( w \) to \( w+h \).

So \( F(w+h) = \int f(z) \, dz \)

\[ \begin{align*}
& \quad \alpha_1 + \alpha_2 + \beta_2 \\
F(w) = & \int f(z) \, dz \\
& \quad \alpha_1 + \beta_1 + \gamma_2
\end{align*} \]

So \( F(w+h) - F(w) = \int f(z) \, dz \)

\[ \begin{align*}
& \quad \alpha_1 + \alpha_2 + \beta_2 \\
& \quad \gamma_1 + \gamma_2 \\
& \quad \beta_3 + \beta_2 \\
& \quad \gamma_2 - \beta_1 + \alpha_2 \\
& \quad \beta_3 \\
& \quad \text{by rectangle lemma}
\end{align*} \]

So \( F(w+h) - F(w) = \frac{f(w)h}{\text{h}} + \varepsilon(h) \), where \( \varepsilon(h) \to 0 \) as \( h \to 0 \).
**Rectangle Lemma**

Let $R$ be a rectangle with diagonal length $D$ and perimeter $P$.

\[ \gamma = \text{join of 4 paths}. \]

We want \( \oint_{\gamma} f(z) \, dz = 0 \).

\[ \oint_{\gamma} f(z) \, dz = \oint_{\gamma_1} f(z) \, dz + \oint_{\gamma_2} f(z) \, dz + \oint_{\gamma_3} f(z) \, dz + \oint_{\gamma_4} f(z) \, dz. \]

\[ |\oint_{\gamma} f(z) \, dz| \leq |\oint_{\gamma_1} f(z) \, dz| + |\oint_{\gamma_2} f(z) \, dz| + |\oint_{\gamma_3} f(z) \, dz| + |\oint_{\gamma_4} f(z) \, dz|. \]

where \( |\oint_{\gamma} f(z) \, dz| \) is the max of the 4 values.

Let \( \gamma_1 = \partial R_1 \) and pick \( \gamma_2 = \partial R_2 \) s.t.

\[ |\oint_{\gamma} f(z) \, dz| \leq 4 |\oint_{\gamma_1} f(z) \, dz|. \]

Induct: \( R \supset R_1 \supset R_2 \supset \ldots \supset R_k \)

\[ |\oint_{\gamma} f(z) \, dz| \leq 4^k |\oint_{\gamma_{k+1}} f(z) \, dz|. \]

Let \( \bigcap_{k=1}^{\infty} \text{cl}(R_k) = \mathbb{Z}_0 \) (analysis!)

(a decreasing intersection of nonempty compact sets $\emptyset$ is itself non-empty)

\( D_k = \text{diam}(R_k) = 2^k D \)

\( P_k = \text{per}(R_k) = 2^k P \)

\( D \) can come with a sequence argument.)
\textbf{punchline}:

\( f \) is analytic at \( z_0 \).

For \( z \) near \( z_0 \)

\[ f(z) = f(z_0) + f'(z_0)(z - z_0) + e(z) \]

\[ \frac{e(z)}{z - z_0} \to 0 \quad \text{as} \quad z \to z_0 \]

Let \( \varepsilon > 0 \).

Pick \( k \) s.t.

\[ |\frac{e(z)}{z - z_0}| < \varepsilon \quad \forall z \in R_k \]

\[ \left| \int_{\delta_k} f(z) \, dz \right| = \left| \int_{\delta_k} f(z_0) \, dz + \int_{\delta_k} f'(z_0)(z - z_0) \, dz + \int_{\delta_k} e(z) \, dz \right| \]

\[ \Rightarrow \quad \int_{\delta_k} f(z) \, dz = 0 \]

\[ \Rightarrow \quad \int_{\delta_k} f(z_0) \, dz = 0 \]

\[ \Rightarrow \quad \int_{\delta_k} e(z) \, dz = 0 \]

\[ \Rightarrow \quad \int_{\delta_k} f(z) \, dz = 0 \]

\[ \Rightarrow \quad \int_{\delta_k} e(z) \, dz = 0 \]

\[ \Rightarrow \quad |\int_{\delta_k} f(z) \, dz| = |\int_{\delta_k} e(z) \, dz| \]

\[ \leq \int_{\delta_k} |e(z)| \, dz \]

\[ \leq \int_{\delta_k} \frac{\varepsilon}{z - z_0} \, dz \]

\[ \leq \int_{\delta_k} \varepsilon |z - z_0| \, dz \]

\[ \leq \int_{\delta_k} \varepsilon D_k \, dz = \varepsilon D_k P_k = \varepsilon 4^{-k} D \]

\textbf{Using} \( \varepsilon \) \textbf{on page 1},

\[ \varepsilon \leq 4^k \varepsilon 4^{-k} D P = \varepsilon D \]

\( \varepsilon \) \textbf{was arbitrary!}

\[ \Rightarrow \quad \left| \int_{\delta_k} f(z) \, dz \right| = 0 \]

\[ \Rightarrow \quad \int_{\delta_k} f(z) \, dz = 0 \]