Math 4200
Friday 9/23

HW for Fri 9/30

§2.3 3, 4, 5, 9, 10
§2.4 2, 3, 4, 7, 8, 9, 12, 16, 17, 18

Where we are:

- If $A$ is any open set and if $f(z)$ (continuous) has an antiderivative $F(z)$ in $A$ then contour integrals $\int_\gamma f(z)dz$ in $A$ are path independent.

- Path independence in a connected open set is equivalent to $\exists F$ s.t. $F'(z)=f(z)$ $\forall z\in A$ (for $\int_\gamma f(z)dz$)

- If $A$ is open and simply connected and $f\in C'(A)$ is analytic, then integrals $\int_\gamma f(z)dz$ are path independent, so also $\exists$ antideriv. $F(z)$ for $f(z)$ (used Green's Thm)

We have proven the 1st two theorems carefully, but the third one only semi-carefully since we didn't precisely understand simply connected, and couldn't show path independence for all paths

The goal of §2.3 is to understand this third theorem and the deformation theorem precisely. (about switching contours w/o changing the value of the contour integral)

Key Step:

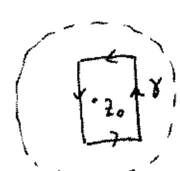
Theorem:
Local Antiderivative (is special case) Let $D(z_0,r)=\{z \mid |z-z_0|<r\}$. Let $f:D(z_0,r)\to\mathbb{C}$ be complex differentiable $\forall z\in D(z_0,r)$. Then $\exists F:D\to\mathbb{C}$ s.t. $F'(z)=f(z)$ $\forall z\in D$.

note: do not need $f\in C^1$ or Green's Thm.

The Key step Theorem will follow from

Lemma: Let $R=[a,b]\times[c,d]$ be any rectangle (sides // to coord dirs) in $D$
let $\gamma=\partial R$ (counterclockwise orientation)

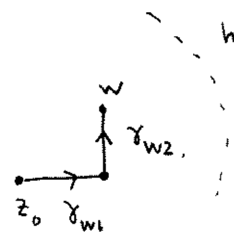
Then $\oint_\gamma f(z)dz = 0$



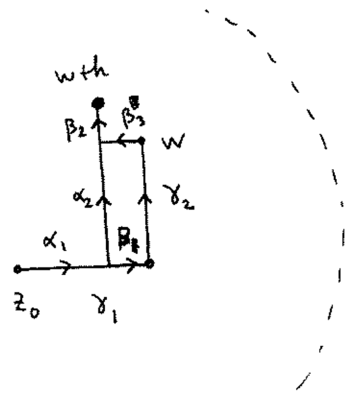
Assuming lemma for now, deduce local antidiff thm:

<u>Define</u> $F(w) = \int_{\gamma_w} f(z)\,dz$ where $\gamma_w = \underset{\text{horiz}}{\gamma_{w1}} + \underset{\text{vert}}{\gamma_{w2}}$

for $w \in D$




Must compare $F(w+h)$ to $F(w)$.




Write $\gamma_w = \gamma_1 + \gamma_2$.

$\gamma_{w+h} = \alpha_1 + \alpha_2 + \beta_2$

with $\alpha_1 + \beta_1 = \gamma_1$

and $\beta_3 + \beta_2$ the horizontal then vertical displacement curve from $w$ to $w+h$

So $F(w+h) = \int_{\alpha_1+\alpha_2+\beta_2} f(z)\,dz$

$F(w) = \int_{\alpha_1+\beta_1+\gamma_2} f(z)\,dz$

So $F(w+h) - F(w) = \int_{\beta_2} f(z)\,dz + \cancel{\int_{\alpha_1-\alpha_1} f(z)\,dz}^{\;0}$

$$+ \underbrace{\int_{\alpha_2} f(z)\,dz - \int_{\beta_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz}_{\int_{-\gamma_2-\beta_1+\alpha_2} f(z)\,dz}$$

$$\int_{-\gamma_2-\beta_1+\alpha_2} f(z)\,dz = \int_{\beta_3} f(z)\,dz \quad \boxed{\text{by rectangle lemma}}$$

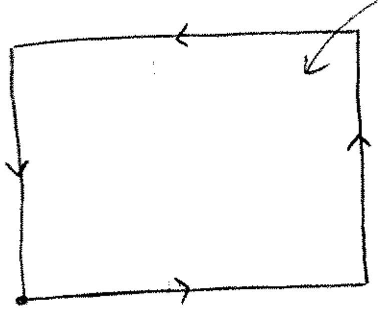
So $F(w+h) - F(w) = \int_{\beta_3+\beta_2} f(z)\,dz$

$$= \int_{\beta_3+\beta_2} f(w) + (f(z)-f(w))\,dz$$

$$= \underbrace{f(w)\int_{\beta_3+\beta_2} dz}_{f(w)h} + \underbrace{\int_{\beta_3+\beta_2} f(z)-f(w)\,dz}_{\left|\quad\right| \le \int |f(z)-f(w)|\,|dz| \le \left(\max_{|z-w|\le|h|} |f(z)-f(w)|\right) 2|h|}$$

So $\dfrac{F(w+h)-F(w)}{h} = f(w) + \varepsilon(h)$, where $\varepsilon(h) \to 0$ as $h \to 0$

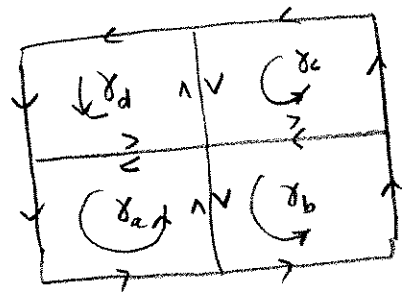
## Rectangle lemma



$R$, diagonal (length) $= D$
perimeter $= P$

$\gamma =$ join of 4 paths.

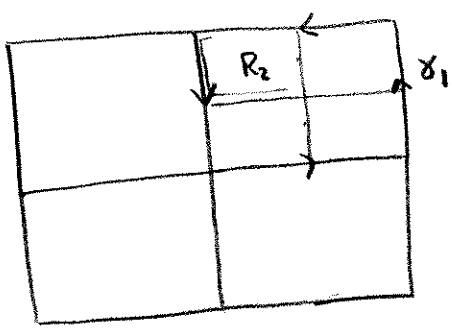
want $\displaystyle\oint_\gamma f(z)dz = 0.$




$$\int_\gamma f(z)dz = \int_{\gamma_a} + \int_{\gamma_b} + \int_{\gamma_c} + \int_{\gamma_d}.$$

$$\left|\int_\gamma f(z)dz\right| \le \left|\int\right| + \left|\int\right| + \left|\int\right| + \left|\int\right| \le 4\left|\int_{\gamma_1} f(z)\right|$$

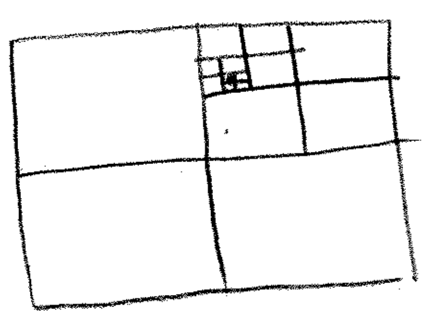
where $\left|\int_{\gamma_1} f(z)dz\right|$ is the max of the 4 values




$\gamma_1 = \partial R_1$

pick $\gamma_2 = \partial R_2$ s.t.

$$\left|\int_{\gamma_1} f(z)dz\right| \le 4\left|\int_{\gamma_2} f(z)dz\right|$$



Induct: $R \supset R_1 \supset R_2 \supset \ldots \supset R_k$

$$\left|\int_\gamma f(z)dz\right| \le 4^k \left|\int_{\gamma_k} f(z)dz\right| \qquad *$$

$$\bigcap_{k=1}^{\infty} cl(R_k) = z_0$$

closure

(analysis!) (a decreasing intersection of nonempty compact sets is itself non-empty ~ can prove with a sequence argument)

$D_k = \text{diag}(R_k) = 2^{-k} D$

$P_k = \text{per}(R_k) = 2^{-k} P$

punchline:

$f$ is analytic at $z_0$.

for $z$ near $z_0$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + e(z) \qquad \left|\frac{e(z)}{z-z_0}\right| \to 0 \text{ as } z \to z_0$$

Let $\varepsilon > 0$. pick $k$ s.t. $\left|\frac{e(z)}{z-z_0}\right| < \varepsilon \quad \forall z \in \overline{R_k}$

$$\left| \int_{\gamma_k} f(z)\,dz \right| = \left| \oint_{\gamma_k} f(z_0)\,dz + \int_{\gamma_k} f'(z_0)(z-z_0) + \int_{\gamma_k} e(z)\,dz \right|$$

antideriv $F = z f(z_0)$ $\Rightarrow \int = 0$

antideriv $F = f'(z_0)\frac{1}{2}(z-z_0)^2$ $\Rightarrow \int = 0$

$$\Rightarrow \left| \int_{\gamma_k} f(z)\,dz \right| = \left| \int_{\gamma_k} e(z)\,dz \right|$$

$$\leq \int_{\gamma_k} |e(z)|\,|dz|$$

$$\leq \int_{\gamma_k} \varepsilon\,|z-z_0|\,|dz|$$

$$\leq \int_{\gamma_k} \varepsilon D_k\,|dz| = \varepsilon D_k P_k = \varepsilon 4^{-k} DP$$

Using $*$ on page 1, get

$$** \quad \left| \int_{\gamma} f(z)\,dz \right| \leq 4^k \varepsilon 4^{-k} DP = \varepsilon DP$$

$\varepsilon$ was arbitrary!

$$\Rightarrow \left| \int_{\gamma} f(z)\,dz \right| = 0$$

$$\Rightarrow \int_{\gamma} f(z)\,dz = 0 \qquad \blacksquare$$