Wed Sept 21
Math 4200 §3.2-3.3
continuing Monday's notes...

- Do "path independence iff antiderivative thm"

- New question: what conditions on A guarantee that if $f: A \to \mathbb{C}$ is analytic, then integrals are path independent?

<u>Theorem</u>: If f is analytic and C^1, and if its open domain A "has no holes"
then integrals are path independent (and $\exists\, F$ s.t. $F'(z) = f(z)$)
In particular, integrals over closed loops in A,
$\int_\gamma f(z)\,dz$, are all zero!

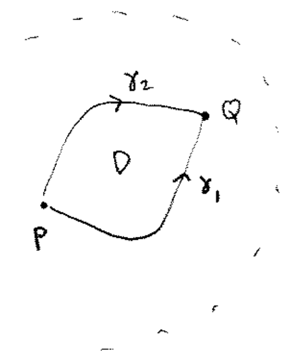
precise condition is that A is <u>simply connected</u>:
(i) A is connected
(ii) every closed path can be continuously deformed to a constant (point) path, with the entire process taking place in A.

We prove the precise version of this Thm on <u>Friday</u>.

<u>proof</u>: This "proof" uses <u>Green's Theorem</u> from 2210.

$$\left(\iint_D Q_x - P_y \, dA = \oint_{\partial D} P\,dx + Q\,dy \right)$$

Case (a)



$$\int_{\gamma_1 - \gamma_2} f(z)\,dz$$

$$= \oint_{\partial D} (u+iv)(dx + i\,dy)$$

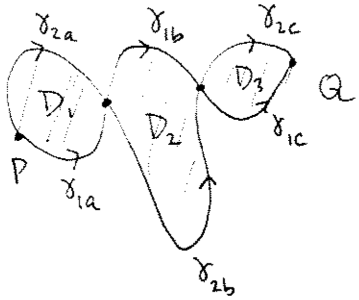
$$= \oint_{\partial D} u\,dx - v\,dy \; + i \oint_{\partial D} v\,dx + u\,dy$$

by Green's: $$= \iint_D (-v)_x - u_y \, dA \; + i \iint_D u_x - v_y \, dA$$

by Cauchy-Riemann! $$= 0 \quad + \quad 0 \quad !!$$

So $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ ∎ for Case (a)

Case (b) more complicated paths:

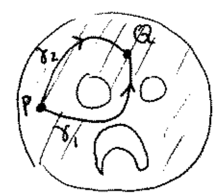


Does $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$?

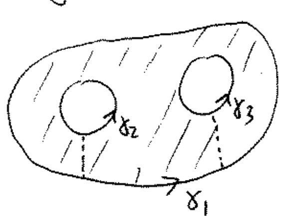
Explain:

Of course, this still doesn't cover the most general case.
But all will be made precise Friday (~Monday).

(c) Notice why required no holes in A for this proof:



Other games we can play:

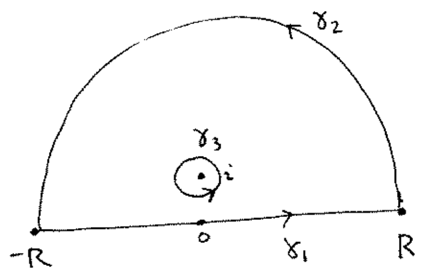


Explain (by creating a contour which does surround a simply connected domain) why, if f is analytic and C^1 in a neighborhood of the shaded region, then

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

Example : Compute $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ w/o using arctan.

Hint : Consider $f(z) = \frac{1}{1+z^2}$



(1) $\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_3} f(z)dz = \int_{\gamma_3} \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) dz$

$=$

(2) Now let $R \to \infty$.

Show $\int_{\gamma_1} f(z)dz \to \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$\int_{\gamma_2} f(z) \to 0$

MAGIC!!