Math 4200
Friday 2 Sept

§1.5 cont'd

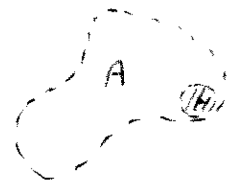
HW for Friday 9/9 ①

- HW 1,2,3,4 in today's notes
- §1.5 1d, 5a, 6b (draw the L-boxes for 5a, 6b) 8, 9, 10, 11, 16, 18abc, 19, 25, 26, 27, 28.

We shall use §1.4 as a reference section for 3220 topics, as needed, and now proceed to §1.5

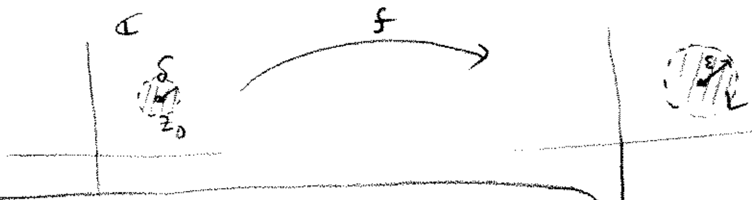
Def For $z_0 \in \mathbb{C}$, $\rho > 0$. $D(z_0, \rho) := \{ z \in \mathbb{C} \text{ s.t. } |z - z_0| < \rho \}$

Def $A \subset \mathbb{C}$ is open iff $\forall z \in A \ \exists \rho > 0$ s.t. $D(z, \rho) \subset A$



Let $A \subset \mathbb{C}$ open
$z_0 \in A$
$f : A \to \mathbb{C}$

Def $\lim_{z \to z_0} f(z) = L$ iff $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. ${}^{0<}|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$



HW1: If $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$, prove

1a) $\lim_{z \to z_0} f(z) + g(z) = L + M$

1b) $\lim_{z \to z_0} f(z) g(z) = LM$

1c) $\lim_{z \to z_0} \frac{1}{g(z)} = \frac{1}{M}$ provided $M \neq 0$

1d) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$ provided $M \neq 0$

Def $A \subset \mathbb{C}$ open, $f : A \to \mathbb{C}$. f is cont at z_0 iff $\lim_{z \to z_0} f(z) = f(z_0)$

Cor of HW 1: sums, products of cont fns are cont.
also quotients if denom $\neq 0$.

Def $A \subset \mathbb{C}$ open
$f: A \to \mathbb{C}$
$z_0 \in A$

f is complex differentiable at z_0 iff
$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} := f'(z_0) \text{ exists.}$$

Note, this limit is the same as
$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

lemma f is differentiable at z_0, with $f'(z_0)=b$ iff there holds an affine approx:

(we did this Monday already)
$$f(z_0+h) = f(z_0) + h\,b + e(h), \quad \text{where } \lim_{h\to 0}\frac{e(h)}{h} = 0.$$

pf: $\Rightarrow$: if $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = b$

then $$\lim_{h\to 0}\Big(\underbrace{\frac{f(z_0+h)-f(z_0)}{h} - b}_{:=\hat{e}(h)}\Big) = 0$$

and $$f(z_0+h) = f(z_0) + hb + \underbrace{h\hat{e}(h)}_{:=e(h)} \qquad \blacksquare$$

$\Leftarrow$: if $f(z_0+h) = f(z_0) + hb + e(h)$
where $\lim_{h\to 0} \frac{e(h)}{h} = 0$

then $$\frac{f(z_0+h)-f(z_0)}{h} = b + \frac{e(h)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = b \qquad \blacksquare$$

Cor f diff at $z_0 \Rightarrow f$ cont at z_0

Theorem f, g diffble at $z_0 \Rightarrow f+g$, fg are, also f/g if $g(z_0)\neq 0$. And
$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$
$$(fg)' = f'g + fg'$$
$$(f/g)' = \frac{f'g - fg'}{g^2}$$

pf e.g. $$(fg)'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)g(z_0+h) - f(z_0)g(z_0)}{h}$$
$$= \lim_{h\to 0} \frac{f(z_0+h)\big(g(z_0+h)-g(z_0)\big)}{h} + \frac{\big(f(z_0+h)-f(z_0)\big)g(z_0)}{h}$$
$$= fg' + f'g \quad \text{by limit thms page 1.}$$

$\boxed{\text{HW2 Prove the quotient rule}}$

Cor : usual differentiation rules for polynomials and rational functions

Chain rule : If f is differentiable at z_0
& g is diffble at $f(z_0)$
then $(g \circ f)'(z_0)$ $\exists$, and equals $g'(f(z_0))f'(z_0)$

pf 1 Use real variables chain rule (3220):

from page 2 & previous notes we know that $f = u + iv$ is $\mathbb{C}$-diffble at $z_0 = x_0 + iy_0$ iff $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is real-diffble at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,
with deriv matrix $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ at x_0,
where $f'(z_0) = a + bi$

HW3 : Apply real vars chain rule
(composition of diffble maps $G \circ F$ is diffble with deriv matrix $D(G \circ F(x_0)) = [DG(F(x_0))][DF(x_0)]$
to deduce $\mathbb{C}$-chain rule above.

pf 2 idea:
$$\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} \underbrace{\frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)}}_{\downarrow\ g'(f(z_0))} \underbrace{\frac{f(z) - f(z_0)}{z - z_0}}_{\downarrow\ f'(z_0)}$$

not rigorous because $f(z) = f(z_0)$ is possible

book's slick fix: Write $f(z_0) = w_0$
define
$$H(w) = \begin{cases} \dfrac{g(w) - g(w_0)}{w - w_0} - g'(w_0) & w \neq w_0 \\ 0 & w = w_0 \end{cases}$$

By hypothesis H is continuous at w_0

Since composition of continuous functions is continuous,
HW4

$$\lim_{z \to z_0} H(f(z)) = H(f(z_0)) = 0$$

For $z \neq z_0$
$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = [H(f(z)) + g'(w_0)]\left[\frac{f(z) - f(z_0)}{z - z_0}\right]$$
check! $f(z) = w_0$, $f(z) \neq w_0$

Now take $\lim_{z \to z_0}$!