Math 4200
Friday 2 Sept

HW 1.6 cont'd

We shall use 1.6 as a reference section for 3220 topics, as needed, and now proceed to 1.6.5.

Def For \( z_0 \in \mathbb{C}, \rho > 0 \), \( D(z_0, \rho) = \{ z \in \mathbb{C} \mid |z - z_0| < \rho \} \)

Def A \subseteq \mathbb{C} is open iff \( \forall z \in A \exists \rho > 0 \text{ st. } D(z, \rho) \subseteq A \)

Let \( A \subseteq \mathbb{C} \) open

\( z_0, \rho \in A \)

\( f : A \to \mathbb{C} \)

\[ \lim_{z \to z_0} f(z) = L \iff \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon \]

**HW 1:** If \( \lim_{z \to z_0} f(z) = L \) and \( \lim_{z \to z_0} g(z) = M \), prove

1. \( \lim_{z \to z_0} f(z) + g(z) = L + M \)
2. \( \lim_{z \to z_0} f(z)g(z) = LM \)
3. \( \lim_{z \to z_0} \frac{1}{g(z)} = \frac{1}{M} \) provided \( M \neq 0 \)
4. \( \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M} \) provided \( M \neq 0 \)

Def \( A \subseteq \mathbb{C} \) open, \( f : A \to \mathbb{C}, \) \( f \) is cont at \( z_0 \) iff \( \lim_{z \to z_0} f(z) = f(z_0) \)

Cor of HW 1: sums, products of cont fms are cont.
also quotients if denom \( \neq 0 \).
Def \( A \subset \mathbb{C} \) open
\[ f: A \to \mathbb{C} \]
\( \exists \ f \) is complex differentiable at \( z_0 \) iff
\[ \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \] exists.

Note, this limit is the same as
\[ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \]

Lemma \( f \) is differentiable at \( z_0 \), with \( f'(z_0) = b \) iff there holds an affine approx:
(we did this Monday already)
\[ f(z_0 + h) = f(z_0) + h \cdot b + e(h), \quad \text{where} \quad \lim_{h \to 0} \frac{e(h)}{h} = 0 \]

pf: \( \Rightarrow \) : if \( \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = b \)
then \( \lim_{h \to 0} \left( \frac{f(z_0 + h) - f(z_0) - b}{h} \right) = 0 \)
\[ = \hat{e}(h) \]
and \( f(z_0 + h) = f(z_0) + h \cdot b + h \cdot \hat{e}(h) \)
\[ : = e(h) \]

\( \Leftarrow \) : if \( f(z_0 + h) = f(z_0) + h \cdot b + e(h) \)
\[ \text{where} \quad \lim_{h \to 0} \frac{e(h)}{h} = 0 \]
then \( \frac{f(z_0 + h) - f(z_0)}{h} = b + \frac{e(h)}{h} \)
\[ \Rightarrow \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = b \]

Cor f diff at \( z_0 \) \( \Rightarrow \) f cont at \( z_0 \)
Theorem \( f, g \) diff at \( z_0 \) \( \Rightarrow \) \( f + g, f \cdot g \) are, also \( f/g \) if \( g(z_0) \neq 0 \). And
\[ (f + g)'(z_0) = f'(z_0) + g'(z_0) \]
\[ (fg)' = f'g + fg' \]
\[ (f/g)' = \frac{f'g - fg'}{g^2} \]

If \( e.g. \) \( (fg)'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h)g(z_0 + h) - f(z_0)g(z_0)}{h} \)
\[ = \lim_{h \to 0} \frac{f(z_0 + h)(g(z_0 + h) - g(z_0)) + (f(z_0 + h) - f(z_0))g(z_0)}{h} \]
\[ = fg' + f'g \] by limit thms page 1.

HW2 Prove the quotient rule
Chain rule: If $f$ is differentiable at $z_0$ & $g$ is differentiable at $f(z_0)$
then $(gof)'(z_0)$ exists and equals $g'(f(z_0))f'(z_0)$.

**pf1:** Use real variables chain rule (3220):
from page 2 & previous notes: we know that $f = uv$ is $C^1$ differentiable at $z_0 = x_0 + iy_0$ iff $F(x, y) = (u(x, y), v(x, y))$ is real-differentiable at $(x_0, y_0)$,
with deriv matrix $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ at $x_0$.

where $F'(z_0) = a + bi$

**HW3:** Apply real vars chain rule (composition of differentiable maps $G \circ F$ is differentiable with deriv matrix $D(G \circ F)(z_0) = [DG(F(z_0))] \cdot [DF(z_0)]$)
to deduce $C$-chain rule above.

**pf2:** Idea: $\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \cdot \frac{f(z) - f(z_0)}{z - z_0}$

not rigorous because $f(z) - f(z_0)$ is possible

book's slick fix: Write $f(z_0) = w_0$
define $H(w) = \begin{cases} \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) & w \neq w_0 \\ 0 & w = w_0 \end{cases}$

By hypothesis $H$ is continuous at $w_0$.

Since composition of continuous functions is continuous, $Hf + f = Hf + f = 0$

$\lim_{z \to z_0} H(f(z)) = H(f(z_0)) = 0$

For $z \neq z_0$, $g(f(z)) - g(f(z_0)) = \left[ H(f(z)) + g'(w_0) \right] \cdot \left[ \frac{f(z) - f(z_0)}{z - z_0} \right]$
check! $f(z_0) = w_0$

Now take $\lim_{z \to z_0} !$