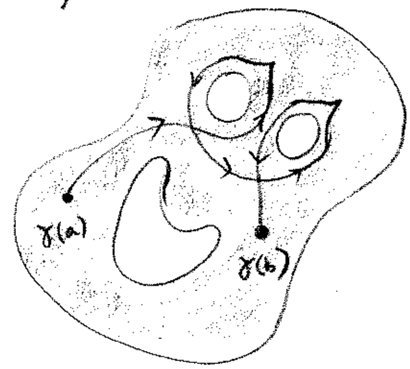
Monday Sept 14
Math 4200
continuing Friday notes...

Magic for computing contour integrals
(when the integrand is analytic)

$\boxed{\text{FTC}}$ If $f(z)$ has a complex antiderivative $F(z)$ in A, and if γ is piecewise C^1 $(\gamma : [a,b] \to A)$

$$\int_\gamma f(z) = F(\gamma(b)) - F(\gamma(a))$$



pf: if γ is C^1 this is on page 3 Wed.
Reproduce argument!!

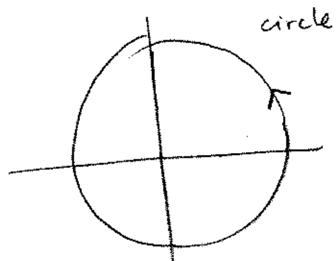
$$\underset{\gamma \quad\;\; = \quad \gamma_1 \;+\; \gamma_2 \;+\; \cdots \;+\; \gamma_n}{}$$

What if $\gamma : [a,b] = [\alpha_0, \alpha_1] \cup [\alpha_1, \alpha_2] \cup \cdots \cup [\alpha_{n-1}, \alpha_n] \to A$ is only piecewise C^1?

pf: (use telescoping series)

$\gamma =$
circle of radius r

Example



What is $\oint_\gamma z^n \, dz$, $n \in \mathbb{Z}$?

try FTC, explicit.
$n = -1$ special.

What conditions on $f : A \to \mathbb{C}$ guarantee that $\exists F$ with $F' = f$
f analytic? conditions on A?

Def $f : \underset{\text{open}}{\overset{\mathbb{C}\,\cup}{A}} \to \mathbb{C}$ continuous, (and assume A is path connected)

contour integrals $\int_\gamma f(z)\,dz$ are <u>path independent</u> (for f)

if they only depend on the endpoints $\gamma(a)$, $\gamma(b)$ of $\gamma : [a,b] \to \mathbb{C}$:

i.e. $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$

$\forall \; \begin{matrix} \gamma_1 : [a,b] \to A \\ \gamma_2 : [c,d] \to A \end{matrix} \qquad \begin{matrix} \gamma_1(a) = \gamma_2(c) \\ \gamma_1(b) = \gamma_2(d) \end{matrix}$

Theorem contour integrals are path independent for f iff $\exists\, F : A \to \mathbb{C}$ s.t. $F'(z) = f(z)$ $\forall z \in A$
(A, f as above)

<u>proof</u>:

$\Leftarrow$: If $F\ \exists$, then path independence follows from FTC see page 6 & Wed notes!

$\Rightarrow$: pick $z_0 \in A$

For $w \in A$

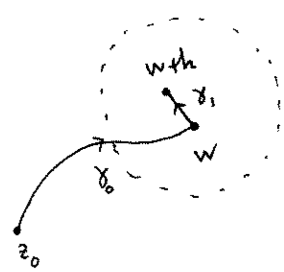
$$F(w) := \int_\gamma f(z)\,dz$$

where γ is (any) piecewise C^1 curve from z_0 to w

since we assume path independence

(F <u>must</u> look like this if it $\exists$)

For $|h|$ small (since A open),
pick $\gamma_0 \subset A$ connecting z_0 + w
and let $\gamma_1(t) = w + th$, $0 \le t \le 1$.



Then

$$F(w+h) = \int_{\gamma_0 + \gamma_1} f(z)\,dz, \quad F(w) = \int_{\gamma_0} f(z)\,dz$$

$$\text{So } F(w+h) - F(w) = \int_{\gamma_1} f(z)\,dz = \int_0^1 \underbrace{f(w+th)}_{=\, f(w) + \varepsilon(th)} h\,dt$$

$\gamma_1(t) = w + th$
$\gamma_1'(t) = h$

where $\varepsilon(th) \to 0$ $\forall\, 0 \le t \le 1$, as $h \to 0$ because f is continuous

$$\text{But } \int_0^1 [f(w) + \varepsilon(th)]h\,dt = f(w)h + \underbrace{\int_0^1 \varepsilon(th)h\,dt}_{|e(h)| := |\ \ | \le \int |\ \ | \le |h| \max\limits_{\substack{|h| \le \delta \\ 0 \le t \le 1}} |\varepsilon(th)|}$$

$\max |\varepsilon(th)| \to 0$ as $h \to 0$

So $F(w+h) = F(w) + f(w)h + e(h)$

where $\frac{e(h)}{h} \to 0$ as $h \to 0$ $\iff F'(w) = f(w)$ ∎