Math 4200 JTB 120 11:50 - 12:40

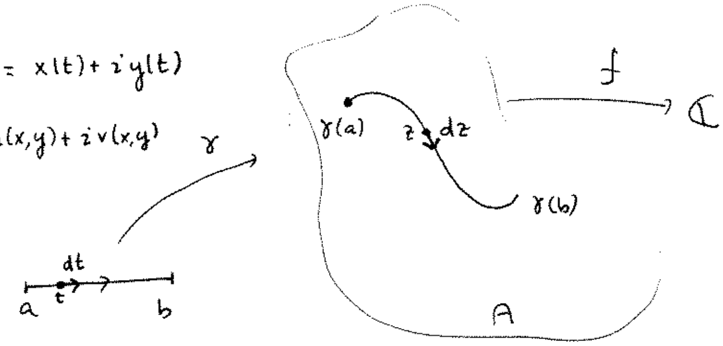
Fri 16 Sept §2.1

Hw for Fri 9/23
2.2 1ad, 2 (prove!) 3, 4, 6, 8, 10, 11
2.3 1, 3, 4, 6, 7

On Wednesday we reviewed line integrals (with real vector fields), and introduced complex fcn path integrals.

Recall: $\gamma : [a,b] \longrightarrow A_{\text{open}} \subset \mathbb{C}$ $\gamma(t) = x(t) + i y(t)$

$f : A \to \mathbb{C}$ $f(z) = u(x,y) + i v(x,y)$



Then

$$* \quad \int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

easy to remember; substitute. $z = \gamma(t)$, $dz = \gamma'(t)\,dt$

We checked

- $\left| \int_\gamma f(z)\,dz \right| \le \int_\gamma |f(z)|\,|dz| := \int_a^b |f(\gamma(t))| \underbrace{|\gamma'(t)|\,dt}_{ds = |dz|}$

- $\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx + i\,dy) = \underbrace{\int_\gamma u\,dx - v\,dy}_{} + i \underbrace{\int_\gamma v\,dx + u\,dy}_{}$

  Math 2210 line integrals

- I also indicated how
  $$\int_\gamma f(z)\,dz = \lim_{\|P\| \to 0} \sum_1 \overbrace{f(\gamma(t_j^*))}^{f(z_j)} \overbrace{(\gamma(t_j) - \gamma(t_{j-1}))}^{\Delta z_j}$$

We can prove this just using * & the chain rule, see today's notes → { which indicates why line integrals like this do no depend on the particular parameterization of the image curve, but only on the direction traversed (sign changes when you traverse γ in the opposite direction)

- If $F'(z) = f(z)$ in A, then $\int_\gamma f(z)\,dz = F(B) - F(A)$ where $B = \gamma(b)$, $A = \gamma(a)$

  F.T.C. for analytic line integrals!

  Do you remember the proof???

What we didn't do enough of on Wed was examples!
I recommend some of your HW that was due today...

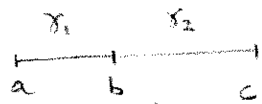
§2.1 p 109-111

11 ab, 13, 14, 10 are all great to discuss.

curve "algebra":

$\gamma_1 : [a,b] \to \mathbb{C}$

$\gamma_2 : [b,c] \to \mathbb{C}$



$\gamma_1 + \gamma_2 : [a,c] \to \mathbb{C}$ (the "join" of γ_1, γ_2)

$$(\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t), & t \in [a,b] \\ \gamma_2(t), & t \in [b,c] \end{cases}$$

& $-\gamma_1 : [a,b] \to \mathbb{C}$

$-\gamma_1(t) := \gamma_1(a+b-t)$

reverses orientation.

piecewise C^1 contours:

If $\gamma : [a,b] = [\alpha_0, \alpha_1] \cup [\alpha_1, \alpha_2] \cup \cdots \cup [\alpha_{n-1}, \alpha_n]$ (with $\alpha_0 = a$, $\alpha_n = b$)

is piecewise C^1, $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, $\gamma_j : [\alpha_{j-1}, \alpha_j] \to \mathbb{C}$ a C^1 curve

$$\int_\gamma f(z)\,dz := \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz$$

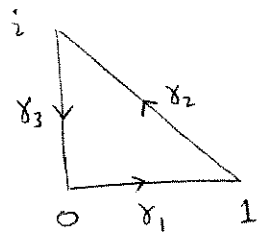
<u>Note</u>: If γ is actually C^1 this def agrees with original one, by additivity properties of "dt" integrals from Calculus.

prop If γ_1, γ_2 are piecewise C^1 then

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

Example



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$$\int_\gamma 1\,dz = \underbrace{\int_{\gamma_1} dz}_{=\,1} + \underbrace{\int_{\gamma_2} dz}_{=\,-1+i} + \underbrace{\int_{\gamma_3} dz}_{=\,-i} = 0$$

because I know what "dz" means

FTC

$$\left(= z\Big]_0^0 = 0 \right)$$

$$\int_\gamma z\,dz = \int_{\gamma_1} z\,dz + \int_{\gamma_2} z\,dz + \int_{\gamma_3} z\,dz$$

FTC

$$\left(= \frac{z^2}{2}\Big]_0^0 = 0 \right)$$

z, dz real >0
$\Rightarrow$ ans >0

$\arg(dz) = \frac{3\pi}{4}$
on average
$\arg(z) = \pi/4$
$\Rightarrow \arg(\text{answer}) = \pi$
ans <0

$\arg(z) = \pi/2$
$\arg(dz) = +3\pi/2$
$\Rightarrow$ ans >0

a conscientious math student

$\gamma_1(t) = t, \quad 0 \le t \le 1$

$\gamma_1'(t) = 1$

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt$$

$$= \int_0^1 t\,dt = \frac{t^2}{2}\Big]_0^1 = \frac{1}{2}$$

a physics student:

$z = x$

$dz = dx$

$0 \le x \le 1$

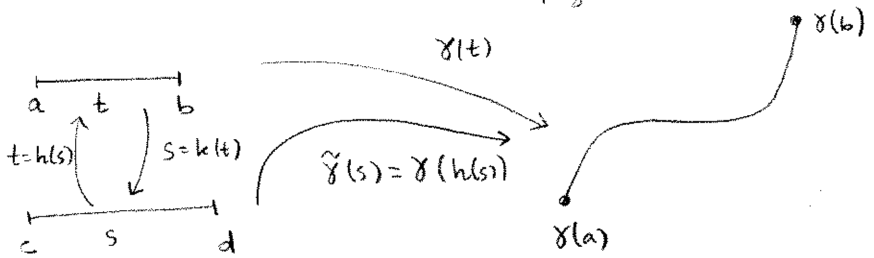
$$\int z\,dz = \int_0^1 x\,dx = \frac{1}{2}$$

doing the same thing in different dialects.

YOU SHOULD FINISH!

try to be a mathematician & a physicist

Reparameterization Theorem (also on page 2 ~~by~~ Wed by partition argument - good intuitively but less precise than this:



$$\int_{\tilde\gamma} f(z)\,dz = \pm \int_{\gamma} f(z)\,dz \ :$$ "+" if $\tilde\gamma$ traverses curve in same dir
"−" if $\tilde\gamma$ reverses direction.

pf. $$\int_{\tilde\gamma} f(z)\,dz := \int_c^d f(\tilde\gamma(s))\,\tilde\gamma'(s)\,ds$$

$$= \int_c^d f(\gamma(h(s)))\,\underbrace{[\gamma(h(s))]'}_{\gamma'(h(s))\,h'(s)}\,ds$$

Subs $t = h(s)$
$dt = h'(s)\,ds$

$$= \int_{h(c)}^{h(d)} f(\gamma(t))\,\gamma'(t)\,dt = \begin{cases} \displaystyle\int_a^b f(\gamma(t))\gamma'(t)\,dt = +\int_\gamma f\,dz & \text{if } h(d)=b,\ h(c)=a \\ \displaystyle\int_b^a f(\gamma(t))\gamma'(t)\,dt = -\int_\gamma f\,dz & \text{if } h(d)=a,\ h(c)=b \end{cases}$$

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