Math 4200
Wednesday 9/14

Chapter 2: Complex integration, leading to the Cauchy Integral formula and some magic corollaries like the fundamental theorem of algebra

Real land

1a)

$u:[a,b] \to \mathbb{R}$ continuous

$$\int_a^b u(t)\,dt := \lim_{\|\text{Part}\| \to 0} \sum u(t_j)\Delta t_j$$

(3210)

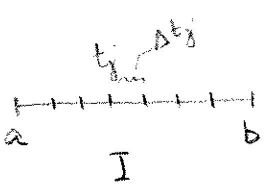
Cor
$$\left|\int_a^b u(t)\,dt\right| = \lim_{\|P\|\to 0} \left|\sum u(t_j)\Delta t_j\right| \le \lim_{\|P\|\to 0} \sum |u(t_j)|\,\Delta t_j \quad (\Delta\text{ ineq!}) = \int_a^b |u(t)|\,dt$$

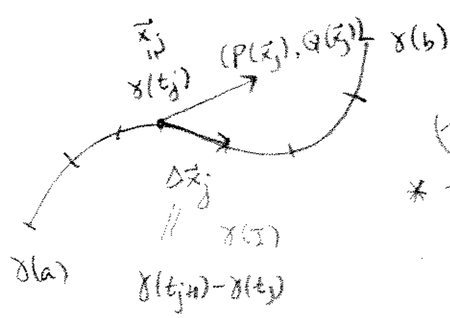
FTC $\int_a^b u(t)\,dt = U(b)-U(a)$
if U is an antiderivative of u

2a) Real line integrals

$\gamma:[a,b] \to A \subset \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$
C^1 diff'ble (or piecewise C^1)
$(P,Q): A \to \mathbb{R}^2$ P, Q continuous

$$\int_\gamma P\,dx + Q\,dy := \int_a^b \left(P(\gamma(t)), Q(\gamma(t))\right) \cdot \underbrace{\left(\tfrac{dx}{dt}, \tfrac{dy}{dt}\right)}_{\vec{\gamma}'(t)} dt \quad *$$





(Thm)
$$* = \lim_{\max|\Delta x_j|\to 0} \sum_j \left(P(\vec{x}_j), Q(\vec{x}_j)\right) \cdot \underbrace{\Delta\vec{x}_j}_{\approx\, \vec{\gamma}'(t_j)\Delta t_j}$$
(work integrals in Physics)

Complex land

1b)

$f:[a,b] \to \mathbb{C}$ continuous

$f(t) = u(t) + i v(t)$

$$\int_a^b f(t)\,dt := \lim_{\|P\|\to 0} \sum f(t_j)\Delta t_j = \lim_{\|P\|\to 0} \left(\sum u(t_j)\Delta t_j + i\sum v(t_j)\Delta t_j\right) = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$
(book's def)

Cor
$$\left|\int_a^b f(t)\,dt\right| = \lim_{\|P\|\to 0}\left|\sum f(t_j)\Delta t_j\right| \le \lim_{\|P\|\to 0} \sum |f(t_j)|\,\Delta t_j = \int_a^b |f(t)|\,dt$$
modulus!

FTC: $\int_a^b f(t)\,dt = F(b) - F(a)$
if $\frac{dF}{dt} = f(t)$ on $[a,b]$

Real land

Facts you remember:

- if you reparameterize your curve in the same direction, you get the same value for the line integral
- if you reverse direction, you get opposite value for line integral

estimate:

$$\left| \int_\gamma P\,dx + Q\,dy \right| = \left| \int_a^b (P,Q)\cdot\gamma'\,dt \right|$$
$$\leq \int_a^b |(P,Q)\cdot\gamma'|\,dt$$
$$\overset{(1a)}{\leq} \int_a^b \|(P,Q)\|\,\|\gamma'\|\,dt$$
$$\overset{C.S.}{=} \int_a^b \|(P,Q)\|\,ds$$

Complex land

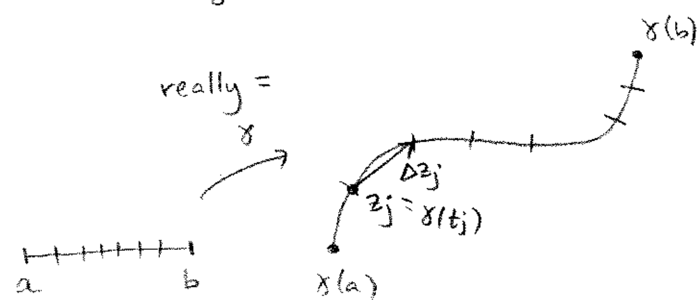
$\gamma : [a,b] \to \underset{\cup}{\overset{\mathbb{C}}{}}A$, $\gamma \in C^1$ (or piecewise C^1),

$\gamma(t) = x(t) + iy(t)$

$f : A \to \mathbb{C}$ continuous

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

really =



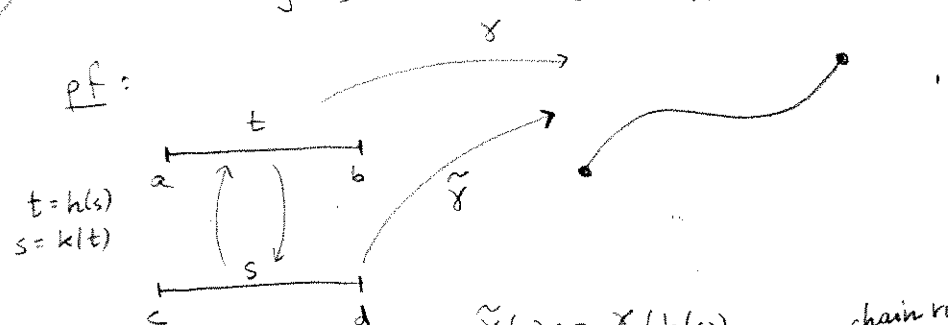
$$\lim_{\max|\Delta z_j| \to 0} \sum f(z_j)\,\Delta z_j$$

$\Delta z_j \approx \gamma'(t_j)\,\Delta t_j$

Facts to check

- reparameterization in same dir gives same value; in opposite dir get opposite value:

pf:



$\tilde\gamma(s) := \gamma(h(s))$

chain rule!

$$\int_{\tilde\gamma} f(z)\,dz = \int_c^d f(\gamma(h(s)))\,\gamma'(h(s))\,h'(s)\,ds$$

subs $t = h(s)$
$dt = h'(s)\,ds$

$$= \int_{h(c)}^{h(d)} f(\gamma(t))\,\gamma'(t)\,dt = \begin{cases} \int_\gamma f\,dz & \text{if } h(c)=a,\ h(d)=b \\ -\int_\gamma f\,dz & \text{if } h(c)=b,\ h(d)=a \end{cases}$$

estimate:

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$\le \int_a^b |f(\gamma(t))|\underbrace{|\gamma'(t)|\,dt}_{\text{(arclength)}} \qquad (1b)$$

$$:= \int_\gamma |f(z)|\,|dz|$$

Connection between
2a), 2b):

$$\int_\gamma f(z)\,dz = \int_\gamma (u(z)+iv(z))(dz)$$

$$\overset{?}{=} \int_\gamma (u(z)+iv(z))(dx+idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx$$

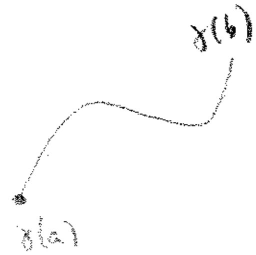
complex line integral

real line integrals

check: $$\int_a^b (u(\gamma(t))+iv(\gamma(t)))\underbrace{\gamma'(t)}_{\frac{dx}{dt}+i\frac{dy}{dt}}\,dt = \int_a^b \left(u(\gamma(t))\frac{dx}{dt} - v(\gamma(t))\frac{dy}{dt}\right)dt + i\int_a^b \left(u(\gamma(t))\frac{dy}{dt} - v(\gamma(t))\frac{dx}{dt}\right)dt$$

∎

"FTC":
If $(P,Q)=\nabla F$
then $$\int_\gamma P\,dx + Q\,dy = F(\gamma(b)) - F(\gamma(a))$$
remember??



If $\exists\, F: A \to \mathbb{C}$, $F \in C^1$
and $F'(z) = f(z)$ (complex deriv.)

then $$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$
(only depends on endpoints!)

proof $$\int_a^b \underbrace{f(\gamma(t))\,\gamma'(t)}_{=\frac{d}{dt}F(\gamma(t))}\,dt$$

"Chain rule for curves".

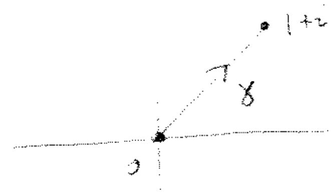
$$\left(= \frac{d}{dt}\left(U(\gamma(t)) + iV(\gamma(t))\right)\right)$$

$$= F(\gamma(b)) - F(\gamma(a))$$

1210 FTC to real & imag parts.

∎

Example



e.g. $\gamma(t) = t(1+i) \quad 0 \le t \le 1$

$dz = (1+i)\,dt$

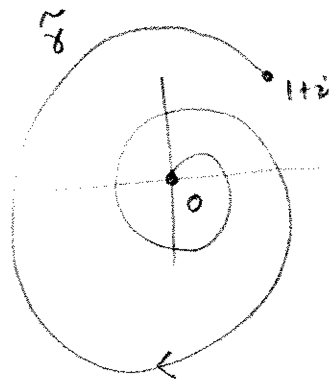
$$\int_\gamma x\,dz = \int_0^1 t(1+i)\,dt = \frac{t^2}{2}(1+i)\Big]_0^1 = \frac{1+i}{2}$$

$$\int_\gamma y\,dz = \int_0^1 (it)(1+i)\,dt = \frac{t^2}{2}(i-1)\Big]_0^1 = \frac{1}{2}(i-1)$$

$$\int_\gamma z\,dz \overset{?}{=} \frac{z^2}{2}\Big]_0^{1+i} = \frac{(1+i)^2}{2} - 0 = i$$

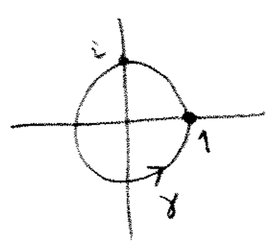
$$= \int_0^1 (t+it)(1+i)\,dt = \frac{1}{2}(1+i) + \frac{1}{2}(i-1)$$

$= i$ ✓



$$\int_{\tilde\gamma} z\,dz = ?$$

<u>very important example:</u>



$\gamma(t) = e^{it} \quad 0 \le t \le 2\pi.$

$$\int_\gamma z^n\,dz = ? \qquad n \in \mathbb{Z}$$

from def

using FTC