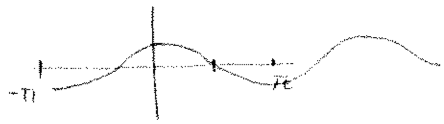
Math 1200
Mon 12 Sept

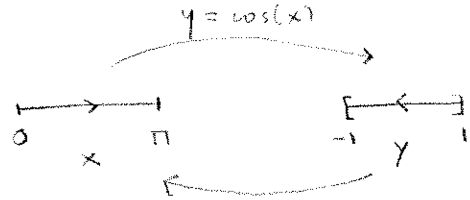
This week only, my MW 1-1:50 office hours are postponed to 2-2:50.

§1.5-1.6: The zoo of common analytic fns & derivatives

example to think about from Calculus:




$y = \cos x$




$x = \cos^{-1}(y) = \arccos(y)$

this is the standard branch (choice) for arccos. If we chose a different domain we would get a different branch of inverse function

$\frac{d}{dy} \quad \cos(\arccos y) = y$

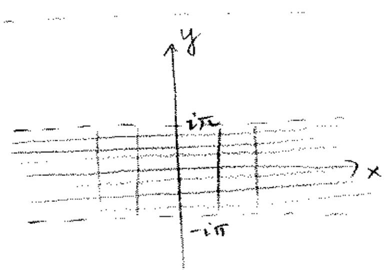
$\Rightarrow -\sin(\arccos y) \cdot \frac{d}{dy} \arccos y = 1$

$\frac{d}{dy} \cos^{-1}(y) = -\frac{1}{\sqrt{1-y^2}}$ if $\cos x = y$, $x \in [0, \pi]$ then $\sin x = \sqrt{1-y^2}$

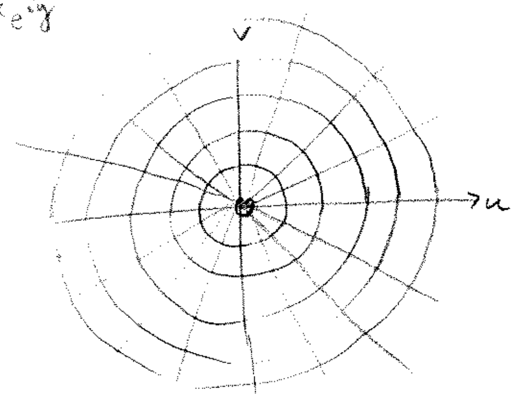
// back to $\mathbb{C}$!

$f(z) = e^z = e^x e^{iy}$

$\boxed{\frac{d}{dz} e^z = e^z}$




$z = x + iy$

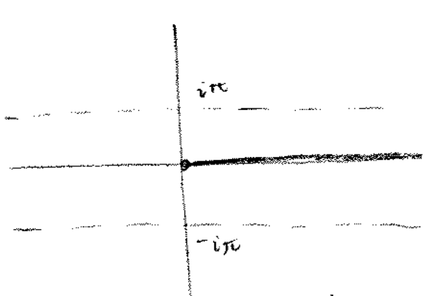


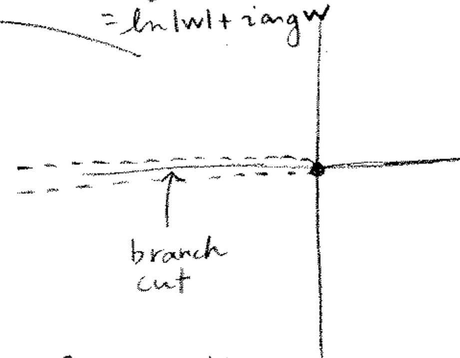

$w = u + iv$

$f^{-1}(w) = g(w) = \log w$
$= \ln|w| + i \arg w$

$\boxed{\frac{d}{dw} \log w = \frac{1}{w}}$

- give 2 proofs:
  - inverse fcn
  - CR



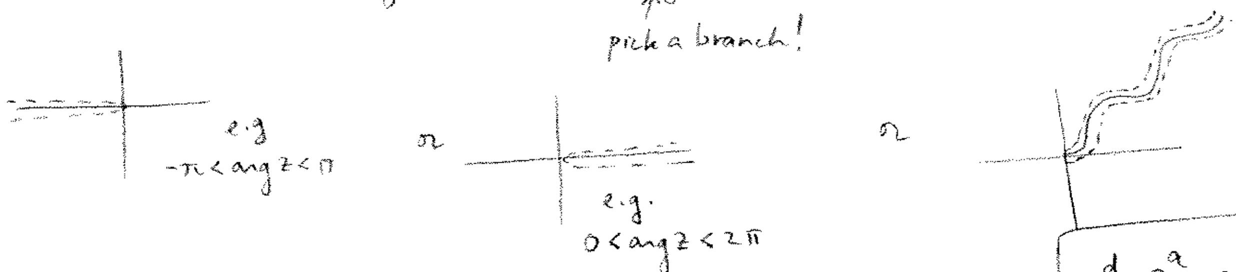

standard branch → often use this choice of log, but not always!

note, on positive x-axis $f, g$ restrict to $e^x$, $\ln x$.

If change domain of exp will change branch of log for any $a \in \mathbb{C}$

$$z^a := e^{a \log z}$$

is composition of exponential $\overset{\text{domain all of } \mathbb{C}}{\downarrow}$ $\omega$ $\underset{\text{pick a branch!}}{\log}$ so has same domain as log choice




$$\boxed{\frac{d}{dz} z^a = a z^{a-1}}$$

$$\frac{d}{dz} z^a = \frac{d}{dz} e^{a \log z} = e^{a \log z}\left(\frac{a}{z}\right) \quad \text{chain rule!}$$

$$= e^{a \log z} a e^{-\log z} = a e^{(\log z)(a-1)} = a z^{a-1}$$

<u>Remarks</u>: (1) if we take standard branch of log the for real positive $x$,

$x^a = e^{a \ln x}$ agrees with Calc def.

(2) if $n \in \mathbb{Z}$ then new def

$$z^n = e^{n \log z} = e^{n(\ln|z| + i \arg z)} = |z|^n e^{in(\theta + 2\pi k)}$$

some choice of arg fn $z = |z|e^{i\theta}$

$$= |z|^n e^{in\theta} = z^n$$

original def, no matter which log branch

(3) if $n \in \mathbb{N}$, $z^{\frac{1}{n}} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + i \arg z)}$

$$= |z|^{1/n} e^{i\left(\frac{1}{n}\arg z\right)}$$

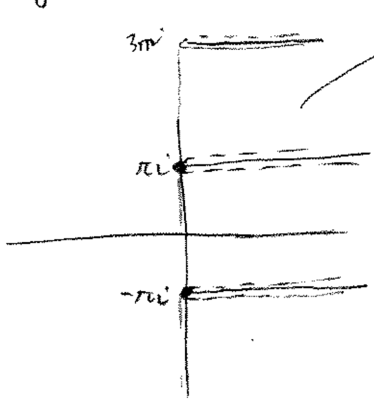
gives all $n^{th}$ roots has we take different choices of arg.

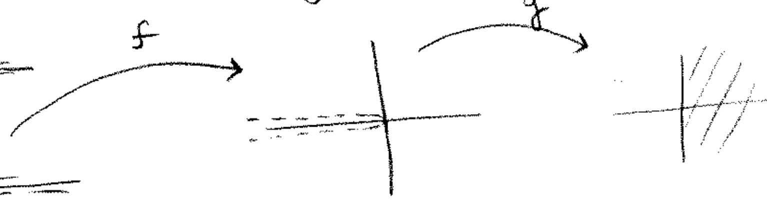
<u>example</u>: find a branch of $\sqrt{e^z+1}$ and compute its derivative.

$$\frac{d}{dz}(e^z+1)^{1/2} = \tfrac{1}{2}(e^z+1)^{-1/2} e^z.$$

$f(z) = e^z + 1$

$g(w) = \sqrt{w}$ — may choose to pick standard branch for $g$




then need $f(z) \notin (-\infty, 0]$ (neg real axis)

$e^z + 1 \notin (-\infty, 0]$

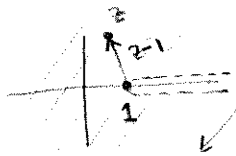
$e^z \notin (-\infty, -1]$

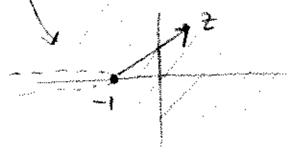
$e^{x+iy} \notin (-\infty, -1]$ leads to domain at left

if $y = \pi + 2n\pi$ need $e^x < 1$, $x < 0$

example $f(z) = \sqrt{z^2-1}$

choose to write as $\sqrt{z-1}\,\sqrt{z+1}$



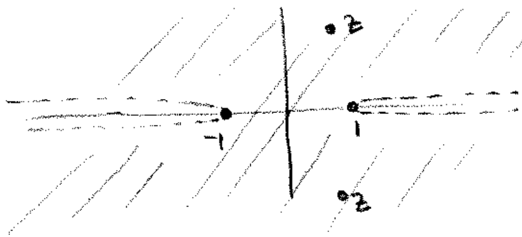

$0 < \arg(z-1) < 2\pi$ $\qquad$ $-\pi < \arg(z+1) < \pi$

another way to get same fcn: use this domain for $\sqrt{\ }$:

$0 < \arg w < 2\pi$

want $z^2 - 1 \notin [0,\infty)$
$z^2 \notin [1,\infty)$

so product is well defined on intersection of domains!




$f(z) = \sqrt{z-1}\,\sqrt{z+1} = |z-1||z+1|\, e^{i\frac{1}{2}(\arg(z-1) + \arg(z+1))}$

remark: image lies in upper half plane:

if $\operatorname{Im} z > 0$

$0 < \arg(z+1) < \pi$
$0 < \arg(z-1) < \pi$
$\Rightarrow 0 < \frac{1}{2}(\arg(z+1) + \arg(z-1)) < \pi$

if $\operatorname{Im} z < 0$

$\begin{cases} -\pi < \arg(z+1) < 0 \\ \pi < \arg(z-1) < 2\pi \end{cases}$

And now we can analyse $\cos^{-1}(w)$!

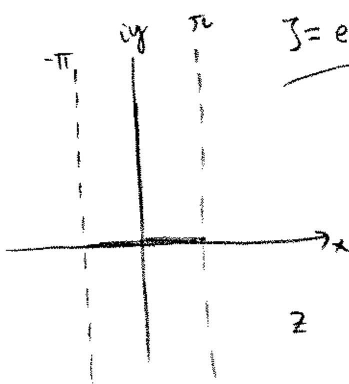
$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

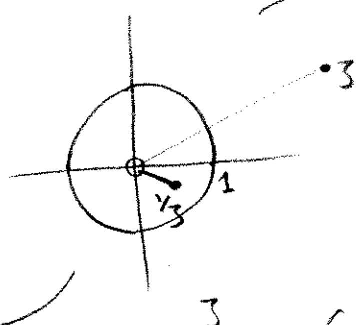
$f(z) = e^{iz} = e^{i(x+iy)} = e^{ix-y}$

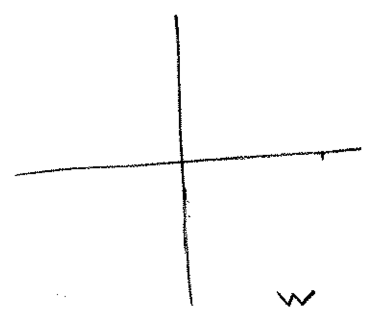
$g(\zeta) = \frac{1}{2}(\zeta + \frac{1}{\zeta})$

$\cos z = g \circ f(z)$

$\zeta = e^{iz}$ $\qquad$ $g(\zeta) = \frac{1}{2}(\zeta + \frac{1}{\zeta})$




$z = -i \log \zeta$

$g$ is 2:1
since if $w = \frac{1}{2}(\zeta + \frac{1}{\zeta})$ then

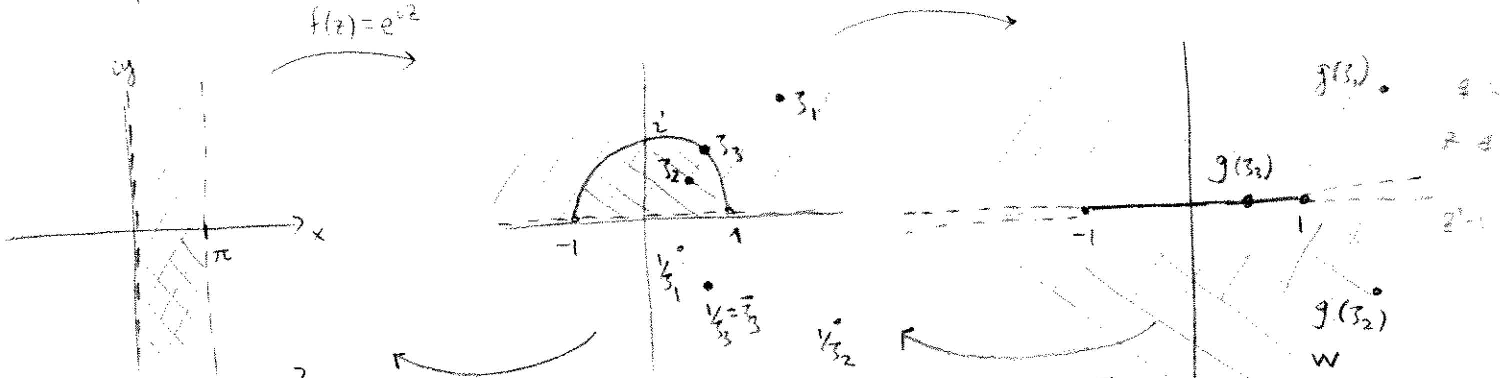
$\zeta^2 - 2w\zeta + 1 = 0$

$\zeta = w + \sqrt{w^2-1}$
(2 values)

in fact
$g(\zeta) = g(\frac{1}{\zeta})$ is clear from formula for $g$

$g$ is 1-1 from upper half $\zeta$-plane, and has inverse
$g^{-1}(w) = w + \sqrt{w^2-1}$
(appropriate branch)

refined picture :




$z=-i\log\zeta$

$\zeta=w+\sqrt{w^2-1}$
using branch
from example on page 3!
e.g. $g^{-1}(0)=+i$

e.g.
$g(e^{i\theta})=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$
$=\cos\theta$

so! for $w=\cos z$

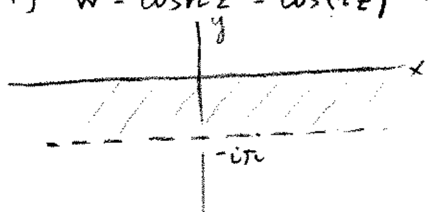
$$\cos^{-1}(w)=-i\log\left(w+\sqrt{w^2-1}\right)\qquad 0<\arg(\cdot)<\pi$$

What if $w$ is real, $-1<w<1$

then
$$\cos^{-1}(w)=-i\log\left(w+i\sqrt{1-w^2}\right)$$
$$=-i\Big[\underbrace{\ln\left(\sqrt{w^2+1-w^2}\right)}_{\ln(1)=0}+i\underbrace{\arg\left(w+i\sqrt{1-w^2}\right)}_{\cos^{-1}w!}\Big]$$
$$=\cos^{-1}(w),$$
the real version!

1, $\sqrt{1-w^2}$, $\theta$, $w$

also, if $w=\cosh z=\cos(iz)$ $\Rightarrow$ $iz=\cos^{-1}(w)$




actually the neg. $\sqrt{\ }$!

$$z=-\log\left(w-\sqrt{w^2-1}\right)$$

if $z=x$ is real, so $w>1$ is real
(on the edge of the branch)

$$\cosh^{-1}(w)=-\ln\left(w-\sqrt{w^2-1}\right)$$
$$=\ln\left(w+\sqrt{w^2-1}\right)\ \text{is correct!}$$