Practice Final exam sol's.
Math 4200 Fall 2005

1. a) f is complex diff'ble at $z_0 \in \mathbb{C}$ $\quad$ ($f: \underset{z_0}{\overset{\Downarrow}{A}} \to \mathbb{C}$, A open)

iff $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} := f'(z_0)$ exists

b) $f(x+iy) = u(x,y) + i\,v(x,y)$ is complex diff'ble at $z_0 = x_0 + iy_0$

iff

$F(x,y) = (u(x,y), v(x,y))$ is real diff'ble at (x_0,y_0) with rotation-dilation deriv. matrix.

In fact, if $f'(z_0) = a+bi$, then the value of the deriv matrix $\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$ at (x_0,y_0) is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

<u>proof</u>:

$f'(z_0)\ \exists,\ = a+bi$

iff $\quad f(z_0+h) = f(z_0) + (a+bi)(h_1+ih_2) + \varepsilon(h)h \qquad h = h_1+ih_2$, $\varepsilon(h) \to 0$ as $h\to 0$

iff $\quad u(x_0+h_1, y_0+h_2) + i\,v(x_0+h_1, y_0+h_2)$

$= u(x_0,y_0) + i\,v(x_0,y_0) + ah_1 - bh_2 + i(ah_2+bh_1) + \varepsilon(h)h$, $\quad \varepsilon(h)\to 0$ as $h \to 0$

iff $\quad \begin{bmatrix} u(x_0+h_1, y_0+h_2) \\ v(x_0+h_1, y_0+h_2) \end{bmatrix} = \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \vec{E}(h_1,h_2)$

(equate real & imag parts)

$\frac{\|E(h_1,h_2)\|}{\|(h_1,h_2)\|} \to 0$ as $\vec{h} \to \vec{0}$.

$\left(\vec{E}(\vec{h}) := \begin{bmatrix} Re(\varepsilon(h)h) \\ Im(\varepsilon(h)h) \end{bmatrix}\right.$

& conversely if $\vec{E} = (E_1, E_2)$ define $\vec{\varepsilon}(h) = \frac{E_1 + iE_2}{h_1+ih_2}$

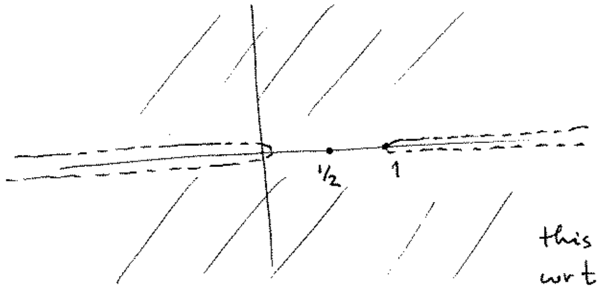
iff $\vec{F}$ is real diff'ble at (x_0,y_0), with deriv. matrix there $= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ ■

2) Several ways to do this:

method 1: $f(z) = \sqrt{z^2 - z} \;:=\; \sqrt{z}\sqrt{z-1}$

where $\sqrt{z} = |z|^{1/2} e^{i(\arg z)/2}$ $\quad -\pi < \arg z < \pi$

$\sqrt{z-1} = |z-1|^{1/2} e^{i(\arg(z-1))/2}$ $\quad 0 < \arg(z-1) < 2\pi$



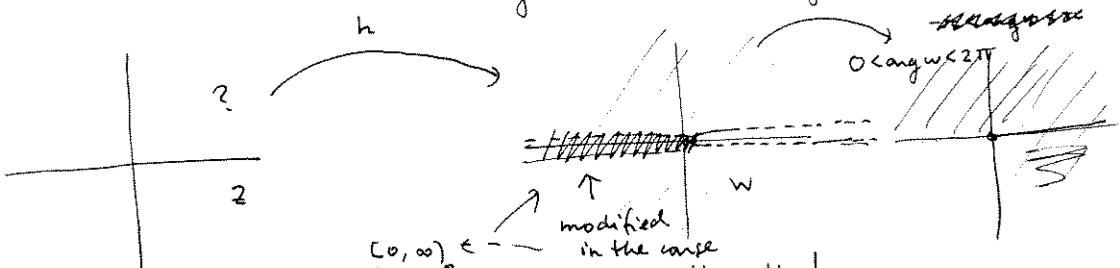
this domain is star-shaped wrt $1/2$, so is simply connected.

method 2: (composition method – works O.K here, for some fns this is the way to go).

$f(z) = g(h(z))$ $\qquad h(z) = z^2 - z = z(z-1)$

$g(w) = w^{1/2}$

$g(w) = \sqrt{w} := |w|^{1/2} e^{i(\arg(w))/2}$



modified in the course of doing the problem!

$h(z) \notin [0,\infty)$

$r^2 e^{2i\theta} - re^{i\theta} \notin [0,\infty)$

$r(r\cos 2\theta + ir\sin 2\theta - \cos\theta - i\sin\theta) \notin [0,\infty)$

$r \neq 0$; need $r\cos 2\theta - \cos\theta < 0$ whenever $r\sin 2\theta - \sin\theta = 0$

$r(2\sin\theta\cos\theta) - \sin\theta = 0$

$r\sin\theta(2\cos\theta - 1) = 0$

$\theta = 0, \pi$

$\theta = 0 \Rightarrow r\cos 2\theta - \cos\theta = r - 1 < 0$

$\boxed{\theta = 0,\ r < 1}$ (eliminate $r \geq 1$)

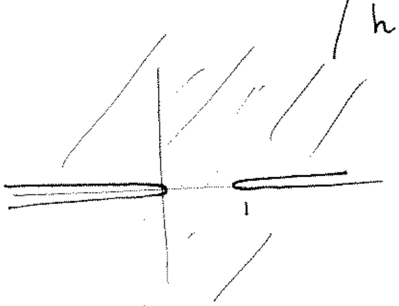
$\theta = \pi \Rightarrow r\cos 2\theta - \cos\theta = r + 1$ is never < 0 (eliminate all r)

$\cos\theta = 1/2 \Rightarrow \theta = \pm \pi/3$

$\Rightarrow 2\theta = \pm 2\pi/3$

$\Rightarrow \cos 2\theta = -1/2$

$\Rightarrow r\cos 2\theta - \cos\theta < 0$ ✓



3a $f(z) = \frac{1}{z^2+2z} = \frac{1}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right)$

for $0 < |z| < 2$, $\frac{1}{z+2} = \frac{1}{2}\left(\frac{1}{1-(-z/2)}\right)$

$= \frac{1}{2}\left(1 + (-\frac{z}{2}) + \frac{z^2}{4} - \frac{z^3}{8} + \dots\right)$

So $f(z) = \frac{1/2}{z} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^n}$ is Laurent in $0<|z|<2$

3b) $\oint_{|z|=1} \frac{1}{z^2+2z}\,dz = \oint_{|z|=1} \frac{1}{2}\frac{1}{z}\,dz + \underbrace{\oint_{|z|=1} \rho(z)\,dz}_{\to\, 0}$ where $\rho(z)$ is analytic in $|z|<2$ so has antideriv.

$= \frac{1}{2}(2\pi i) = \pi i$

3c) $\oint_{|z|=1} \frac{1}{z(z+2)}\,dz = 2\pi i\,(\mathrm{Res}(f,0)) = 2\pi i\left(\frac{1}{2}\right) = \pi i$ (residue from Laurent series! or from simple pole table)

4a) A open & connected is simply connected iff every closed curve is homotopic to a point in A, as closed curves in A.

b) If A is open & simply connected, $f: A \to \mathbb{C}$ analytic in A then $\int_\gamma f(z)\,dz = 0$ $\forall$ (piecewise C^1) closed curves γ in A, by the deformation thm!

So, if $\mathbb{C}\setminus\{0\}$ was simply connected then $\oint_{|z|=1} \frac{1}{z}\,dz$ would equal 0

i.e. $\gamma(t) = e^{it}$ $0 \le t \le 2\pi$

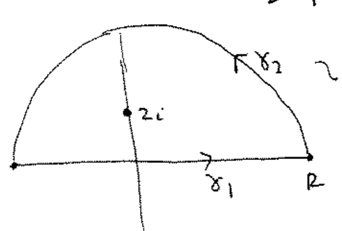
but this integral $= 2\pi i$.

Thus $\mathbb{C}\setminus\{0\}$ is not simply connected.

5) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\,dx$

$= \mathrm{Re}\lim_{R\to\infty} \underbrace{\int_{-R}^{R} \frac{e^{ix}}{x^2+4}\,dx}_{= \int_{\gamma_1} f(z)\,dz}$

$f(z) = \frac{e^{iz}}{z^2+4}$, $\gamma_1(t) = t$ $-R \le t \le R$.



$\gamma_2(t) = Re^{it}$ $0 \le t \le \pi$

$\int_{\gamma_1+\gamma_2} f(z)\,dz = 2\pi i\left(\mathrm{Res}\left(\frac{e^{iz}}{z^2+4}, 2i\right)\right)$

$\frac{e^{iz}}{(z-2i)(z+2i)} = \frac{1}{z-2i}\,\phi(z)$

$\phi(z)$ analytic near $2i$

so $\mathrm{res}(f, 2i) = \phi(2i)$

$= e^{-2}/4i$

$= \frac{2\pi i}{4i} e^{-2} = \frac{\pi}{2} e^{-2}$

$$* \quad \int_{\gamma_1} \frac{e^{iz}}{z^2+4}\,dz + \int_{\gamma_2} \frac{e^{iz}}{z^2+4}\,dz = \frac{\pi}{2}e^{-2} \qquad (R>2).$$

$$\left|\underbrace{\int_{\gamma_2}}\right| \le \int_{\gamma_2} 1 \; \frac{1}{|z^2+4|}\,|dz| \le \int_{\gamma_2} \frac{1}{R^2-4}\,|dz| \quad (R>2)$$

$$= \frac{\pi R}{R^2-4} \to 0 \text{ as } R\to\infty$$

$e^{iz} = e^{i(x+iy)} = e^{ix-y}$

$|e^{iz}| \le 1$ for $y \geq 0$

$|z^2+4| \geq |z|^2-4$ (reverse Δ)

$\frac{1}{|z^2+4|} \le \frac{1}{|z|^2-4}$ $(|z|^2>4)$

$$\lim_{R\to\infty} * \Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\,dx = \frac{\pi}{2}e^{-2}$$

take real part to deduce $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,dx = \frac{\pi}{2}e^{-2}$

6) $\displaystyle\underbrace{\int_0^{2\pi} (\cos(\theta))^6\,d\theta}_{=I}$ convert back to contour integral.

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$; $d\theta = \frac{dz}{iz}$

$\left.\begin{array}{l} z = \cos\theta + i\sin\theta \\ \text{on } |z|=1, \quad \frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta \end{array}\right\} \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\text{so } I = \oint_{|z|=1} \left[\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)\right]^6 \frac{dz}{iz} = 2\pi i \operatorname{Res}\left(\left[\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)\right]^6 \frac{1}{iz}, 0\right)$$

$\frac{1}{2^6}\binom{6}{3} z^3 \left(\frac{1}{z}\right)^3 \frac{1}{iz}$ + terms with net power $\neq -1$

$$= \frac{2\pi i}{2^6 i}\binom{6}{3} = \frac{\pi}{32} \frac{6\cdot5\cdot4}{6\cdot6} = \frac{5\pi}{48}$$

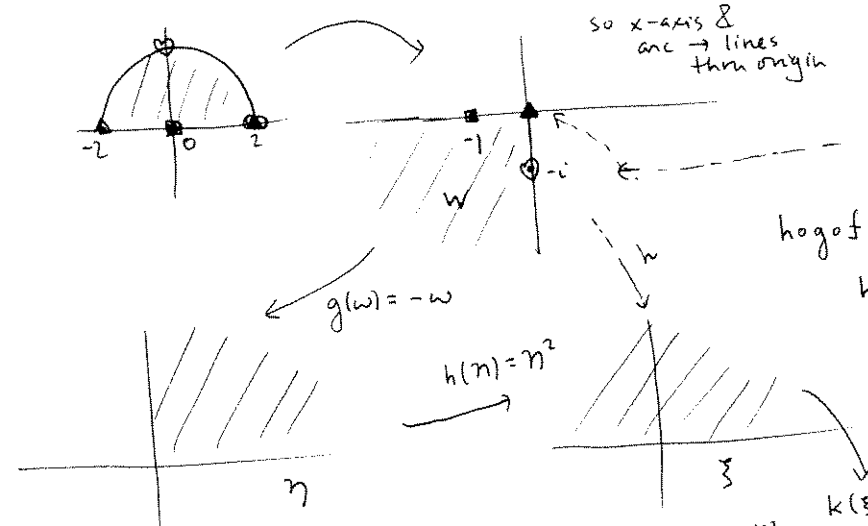
7a) $|z|<2$, $\operatorname{Im}(z)>0$

$f(z) = \frac{z+2}{z-2}$ maps -2 to zero, $+2$ to ∞.

so x-axis & arc $\to$ lines thru origin

$f'(z) = \frac{(z-2)-(z+2)}{(z-2)^2} = \frac{-4}{(z-2)^2}$

$f'(-2) = \frac{-4}{16} = -\frac{1}{16}$



rotates tang vectors by π

$h\circ g\circ f = h\left(\frac{z+2}{2-z}\right) = \left(\frac{z+2}{z-2}\right)^2 = h\circ f$, actually

$h\circ g\circ f\,(i) = \left(\frac{i+2}{i-2}\right)^2 = \left(\frac{i+2}{i-2}\right)^2\left(\frac{-i-2}{-i-2}\right)^2$

$= \frac{(-3-4i)^2}{(5)^2}$

$= \frac{-7+24i}{25}$

$k(\xi) = \left(\xi + \frac{7}{25}\right)\left(\frac{25}{24}\right)$

$F(z) = k\circ h\circ f(z) = \frac{25}{24}\left(\left(\frac{z+2}{z-2}\right)^2 + \frac{7}{25}\right)$ will do!

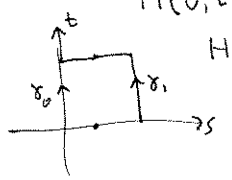
7b) Map not unique. By RMT you can specify $F(z_0)$ and $\arg F'(z_0)$, then you get a unique conformal transformation!

8) γ_0, γ_1 are homotopic as closed curves in A if (after reparameterizing ~~by rescaling~~ γ_0, γ_1 in an orientation preserving way so that they have common t-domain, which we may take to be $[0,1]$)

$\exists\ H : \{[s,t],\ 0 \le s \le 1,\ 0 \le t \le 1\} \to A$, H continuous

$H(0,t) = \gamma_0(t)$, $0 \le t \le 1$ & $H(s,0) = H(s,1)\ \forall\ 0 \le s \le 1$

$H(1,t) = \gamma_1(t)$, $0 \le t \le 1$



8b) $\gamma_0(t) = e^{it} + 2e^{6it}$

$\gamma_1(t) = e^{6it}$

try $H(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$, $0 \le s \le 1$, $0 \le t \le 2\pi$ ($\leftarrow$ could be rescaled to $0 \le t \le 1$)

$H(0,t) = \gamma_0(t)$

$H(1,t) = \gamma_1(t)$

$H(s,0) = (1-s)\gamma_0(0) + s\gamma_1(0)$

$H(s,2\pi) = (1-s)\gamma_0(2\pi) + s\gamma_1(2\pi)$) equal since γ_0, γ_1 are closed.

need $H : [0,1] \times [0,2\pi] \to \mathbb{C} \setminus \{0\}$

i.e. $|H(s,t)| > 0$; $H(s,t) = (1-s)\left[e^{it} + 2e^{6it}\right] + se^{6it}$

$= e^{6it}(\underbrace{2(1-s)+s}_{2-s}) + e^{it}(1-s)$

$|H(s,t)| \geq (2-s) - (1-s)$ reverse Δ ineq $(0 \le s \le 1)$

$= 1$, so $|H(s,t)| \geq 1$, so $H(s,t) \neq 0$ $\quad 0 \le s \le 1$ ∎

8c) $\displaystyle\int_{\gamma_0} \frac{3}{z}\,dz = \int_{\gamma_1} \frac{3}{z}\,dz$ by deformation thm

$\displaystyle = \int_0^{2\pi} \frac{3}{\cancel{e^{6it}}}\, 6i\cancel{e^{6it}}\,dt = 18i \cdot 2\pi = \boxed{36\pi i}$

$z = e^{6it}$

$dz = 6ie^{6it}\,dt$

9) a) if $f : \mathbb{C} \to D(0,1)$ is analytic then Liouville $\Rightarrow$ f is constant so cannot be conformal

b) Let $f : \mathbb{C} \to \mathbb{C}$ conf bijection, $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z$

$\Rightarrow f(\frac{1}{z})$ is analytic except at $z=0$ & $f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n (\frac{1}{z^n})$

If sing at $z=0$ ~~then~~ is essential then Picard $\Rightarrow f(\frac{1}{z})$ is not 1-1 (since every value except 1 is attained ∞'ly often). But $f(\frac{1}{z})$ is 1-1 so $z=0$ is a pole

$\Rightarrow f(\frac{1}{z}) = \sum_{n=0}^{N} a_n (\frac{1}{z^n})$, $a_N \neq 0$

$\Rightarrow f(z) = \sum_{n=0}^{N} a_n z^n$, $a_N \neq 0$

$N=0 \Rightarrow f$ const

$N \geq 2 \Rightarrow f'(z)$ has roots $\Rightarrow f$ not conformal

So $N=1$ $f(z) = az+b$ is a conf shift ∎