Math 4200
Exam 2 Sol'ns

1a) Let f analytic in A, A open

Let $\gamma: I \to A$ p.w. C', γ homotopic to a point in A

Then for $z \in A$, $z \notin \gamma$

$$f(z) I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Here $I(\gamma; z) := \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\, d\zeta$ is the index (or winding number) of γ w.r.t. z

proof Define $F(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$

Since F is analytic in $A \setminus \{z\}$ and continuous at z it is analytic in A [by the modified rectangle lemma]

γ homotopic to a point

$\Rightarrow \int_\gamma F(\zeta)\, d\zeta = 0$ by Deformation Thm

$$z \notin \gamma \Rightarrow \int_\gamma \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta = 0$$

$$= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\underbrace{\int_\gamma \frac{1}{\zeta - z}\, d\zeta}_{2\pi i\, I(\gamma; z)} = 0$$

solve for $f(z)$ in above eqn $\Rightarrow$ C.I.F.

(b) $f^{(n)}(z) I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\frac{d^n}{dz^n}\left(\frac{1}{\zeta - z}\right) d\zeta$

$$= \frac{1}{2\pi i}\, n! \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

(differentiating thru integral can be justified $\Rightarrow$ careful analysis)

2) Pick r s.t. $\overline{D(z_0, r)} \subseteq$ domain of analyticity for f

Let $\gamma(t) = z_0 + re^{it}$ $0 \le t \le 2\pi$

Let $|z - z_0| < r$ so $I(\gamma; z) = 1$

C.I.F.

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$* = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\left(\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}\right) d\zeta$$

$w := \frac{z - z_0}{\zeta - z_0}$ $|w| = \frac{|z - z_0|}{r} < 1$

$$\Rightarrow \frac{1}{1-w} = \sum_{n=0}^\infty w^n$$

$$* = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^\infty \left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \lim_{N\to\infty}\left[\sum_{n=0}^N \frac{f(\zeta)}{\zeta - z_0}\left(\frac{z - z_0}{\zeta - z_0}\right)^n\right] d\zeta$$

Because

$S_N(\zeta) = \sum_{n=0}^N \frac{f(\zeta)}{\zeta - z_0}\left(\frac{z - z_0}{\zeta - z_0}\right)^n \longrightarrow$

$\downarrow$

$S(\zeta) = \sum_{n=0}^\infty (\text{''})$

and convergence is uniform on γ:

$|S(\zeta) - S_N(\zeta)| \le \sum_{n=N+1}^\infty \frac{\max_\gamma |f(\zeta)|}{(r - \rho)}\left(\frac{\rho}{r}\right)^n$

$\rho := |z_0|$

$= \left(\frac{\max_\gamma |f(\zeta)|}{(r - \rho)}\right)\frac{\left(\frac{\rho}{r}\right)^N}{1 - \rho/r} \to 0$ as $N \to \infty$

$$= \lim_{N\to\infty} \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^N \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \lim_{N\to\infty} \sum_{n=0}^N a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^\infty a_n (z - z_0)^n$$

Note, $a_n = \frac{f^{(n)}(z_0)}{n!}$

3a) $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

simple pole at $z=0$ since $\left.\frac{d}{dz}\sin z\right|_{z=0} \neq 0$.

so $z\cot z = a_0 + a_1 z + a_2 z^2 + \dots$

$$z\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\right) = (\overset{=1}{a_0} + \overset{0}{\cancel{a_1 z}} + \overset{-1/3}{a_2} z^2 + \dots)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$z: \; 1 = a_0$

$z^2: \; 0 = a_1$

$z^3: \; -\frac{1}{2} = -\frac{1}{6} + a_2 \Rightarrow a_2 = -\frac{1}{3}$

$z^4: \; 0 = a_3$ (in fact all $a_n = 0$ for n odd since $z\cot z$ is even fn)

$z^5: \; \frac{1}{24} = a_4 + \frac{1}{120} + \frac{1}{18} \Rightarrow a_4 = \frac{15-3-20}{360} = \frac{-8}{360} = -\frac{1}{45}$

$$\boxed{\Rightarrow \cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \dots}$$

3b) $\cot z$ is analytic in the annulus $0<|z|<\pi$

(can't take a larger outside radius because $\sin z = 0$ at $\pm\pi$)

so this is the annulus on which the Laurent series converges

4a) $\displaystyle\oint_{|z|=2} \frac{2z^2}{z^3-1}\,dz$ $\quad *$ $\qquad$ simple poles at $z = e^{\pm\frac{2}{3}\pi i}, 1$

$\mathrm{res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$ if $f(z) = \dfrac{g(z)}{h(z)}$ $\quad g(z_0)\neq 0$, $h(z_0)=0$, $h'(z_0)\neq 0$

here take

$g(z) = 2z^2$

$h(z) = z^3-1 \qquad h'(z) = 3z^2$

Res thm $\Rightarrow * = 2\pi i\left(\mathrm{res}(f;1) + \mathrm{res}(f; e^{\frac{2}{3}\pi i}) + \mathrm{res}(f; e^{-\frac{2}{3}\pi i})\right)$

since $I(\gamma; z_j) = +1 \;\; \forall z_j$ poles

$$\Rightarrow * = 2\pi i\left[\frac{2}{3} + \frac{2\cancel{e^{\frac{4}{3}\pi i}}}{3\cancel{e^{\frac{4}{3}\pi i}}} + \frac{2\cancel{e^{-\frac{4}{3}\pi i}}}{3\cancel{e^{-\frac{4}{3}\pi i}}}\right]$$

$$\boxed{= 4\pi i}$$

4b) $\zeta = \frac{1}{z}$

$z = \frac{1}{\zeta}$

$dz = -\frac{1}{\zeta^2}\,d\zeta$

$$* = \oint_{|\zeta|=1/2} \frac{2\left(\frac{1}{\zeta}\right)^2}{\left(\frac{1}{\zeta}\right)^3-1} \cdot -\frac{1}{\zeta^2}\,d\zeta$$

(clockwise)

$$= --\oint_{|\zeta|=1/2} \frac{2}{\zeta-\zeta^4}\,d\zeta$$

(counterclockwise)

$$= \oint_{|\zeta|=1/2} \frac{2}{\zeta}\frac{1}{(1-\zeta^3)}\,d\zeta$$

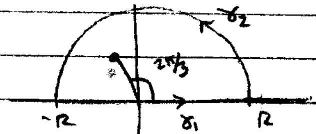
only pole inside circle at $\zeta=0$ is simple;

$\mathrm{res}(f;0) = 2$ since

$f(\zeta) = \frac{1}{\zeta}\left[\frac{2}{1-\zeta^3}\right]$

$$\boxed{\Rightarrow * = 2\pi i\cdot 2 = 4\pi i}$$ ✓

5) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx \;=\; *$



Let $\gamma = \gamma_1 + \gamma_2$:

$f(z) = \frac{1}{z^2+z+1}$

$$\int_\gamma f(z)\,dz = 2\pi i \left[\operatorname{res}(f, e^{2\pi i/3})\right]$$

note $z^3 - 1 = (z-1)(z^2+z+1)$

so poles of f are $e^{\pm 2\pi i/3}$

$f(z) = \frac{g(z)}{h(z)}$ simple pole $\quad g(z) = 1$, $h(z) = z^2+z+1$, $h'(z) = 2z+1$

$$\operatorname{res}(f, e^{2\pi i/3}) = \frac{1}{2e^{2\pi i/3}+1}$$

$$\Rightarrow \int_\gamma f(z)\,dz = \frac{2\pi i}{2e^{2\pi i/3}+1} = \frac{(2\pi i)}{2(\underbrace{\cos\tfrac{2\pi}{3}}_{-1/2} + i\underbrace{\sin\tfrac{2\pi}{3}}_{\sqrt{3}/2}) + 1}$$

$$= \frac{2\pi i}{\sqrt{3}\, i} = \frac{2\pi}{\sqrt{3}}$$

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)\,dz \right| \le \lim_{R\to\infty} \pi R \left(\frac{1}{R^2 - R - 1} \right) = 0$$

$$\lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz = *$$

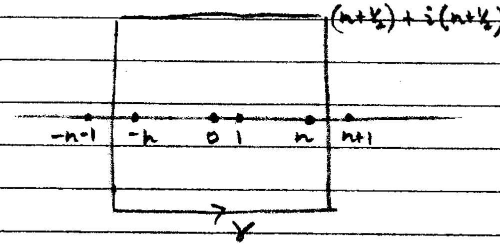
$$\Rightarrow \frac{2\pi}{\sqrt{3}} = \lim_{R\to\infty} \int_\gamma f(z)\,dz = \lim_{R\to\infty} \int_{\gamma_1+\gamma_2} f(z)\,dz = 0 + *$$

$$\boxed{\text{So } * = \frac{2\pi}{\sqrt{3}}}$$

6) $f(z) = \frac{1}{(z-\frac{1}{2})^2}$

then $$\sum_{\substack{n\in\mathbb{Z} \\ n \text{ not sing of } f}} f(n) + \sum_{\substack{z_j \text{ sing} \\ \text{of } f}} \operatorname{res}(f(z)\pi\cot\pi z, z_j) = 0$$

(this was obtained by considering contours γ:



we showed $|\pi\cot\pi z| \le C$ on each such γ,

so if $|f(z)| \le \frac{C'}{|z|^p}$ $\quad p > 1$ as $z \to \infty$

then $\left| \int_\gamma f(z)\,\pi\cot\pi z \right| \le \frac{C'C}{(n+\frac{1}{2})^p}\, 4(2n+1) \to 0$ as $n\to\infty$.

The result then follows from the residue thm.

$\operatorname{res}\left(\frac{1}{(z-\frac{1}{2})^2}\pi\cot\pi z;\ \frac{1}{2}\right)$

double pole, so want $(\pi\cot\pi z)'(z_0)$

$$= -\pi^2\csc^2\pi z\Big|_{z_0} = \frac{-\pi^2}{\sin^2\pi/2} = -\pi^2$$

$$\boxed{\Rightarrow \sum_{n\in\mathbb{Z}} \frac{1}{(n-\frac{1}{2})^2} = \pi^2}$$

b) by trickery:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-\frac{1}{2})^2} = \sum_{n=-\infty}^{\infty} \frac{4}{(2n-1)^2} = 4\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=0}^{-\infty}\frac{1}{(2n-1)^2}\right]$$

$$= 8 \sum_{m \text{ odd}} \frac{1}{m^2}$$

Now, from text,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n \text{ even}} \frac{1}{n^2} = \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{24}$$

$$\Rightarrow \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{(n-\frac{1}{2})^2} = 8\left(\frac{\pi^2}{8}\right) = \pi^2 \checkmark$$