Math 4200 (1)

Monday 10/31

- Residue theorem when $\gamma = \partial D$ (page 2 Fri.)
- computing residues. (see also Fri. p 3-4)
  See table p. 250

simple pole tests 3-5 easy. (See e.g. p. 4 Fri)

example: Compute the residues of

$$f(z) = k(z)\pi \cot(\pi z) \qquad \text{at } z = n \in \mathbb{Z}, \text{ assuming } k(n) \neq 0$$

f has simple poles at n since $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$:

$k(n)\cos \pi n \neq 0$

$\sin \pi n = 0$

$\dfrac{d}{dz} \sin \pi z \Big|_{z=n} \neq 0.$

$$\text{Res}\left(\frac{k(z)\pi \cos \pi z}{\sin \pi z} ; n \right)$$

$$= \frac{k(n)\pi \cos \pi n}{\pi \cos \pi n} \quad (4)$$

$$= k(n).$$

This will let us sum interesting series (e.g. $k(z) = \frac{1}{z^2}$) with the residue theorem - see §4.4.

example $\text{Res}\left(\dfrac{z^2}{(z-1)^2(z+1)} ; -1 \right)$

$$\frac{z^2/(z-1)^2}{(z+1)} = \frac{a_0 + a_1(z+1) + \cdots}{(z+1)}$$

$$a_0 = (-1)^2/(-2)^2 = \frac{1}{4}$$

double pole tests:

(7) easy (too),

(6) instructive (8 similar)

see Fri p.4

$$f(z) = \frac{g(z)}{h(z)} \qquad g(z_0) \neq 0, \quad h(z_0) = h'(z_0) = 0 \qquad \text{Res}(f; z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$$

example $\text{Res}\left(\dfrac{z^2}{(z-1)^2(z+1)} , 1 \right)$

$g = z^2$ $\qquad g' = 2z$

$h = (z-1)^2(z+1)$

$= (z^2-1)(z-1)$

$= z^3 - z^2 - z + 1$

$h' = 3z^2 - 2z - 1$

$h'' = 6z - 2$

$h''' = 6$

$$\text{Res} = \frac{2 \cdot 2}{4} - \frac{2}{3}\,\frac{1 \cdot 6}{4^2}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

or use (7)

Higher order poles : same ideas. (see also next page!)

1. if $f(z)$ has a pole of order k at z_0

   then $(z-z_0)^k f(z)$ is analytic there (after removing sing.)

   $$\underset{\phi(z)}{\underbrace{(z-z_0)^k f(z)}}$$

   want coeff of $(z-z_0)^{k-1}$ in Taylor for ϕ,

   i.e. $$\mathrm{Res}(f, z_0) = \frac{1}{(k-1)!}\phi^{(k)}(z_0)$$

2. Use power series and the multiply through trick

3. resort to technology!

   $$\mathrm{residue}\left(\frac{e^z}{(\sin z)^5}; 0\right)$$

   pole of order 5!

Maple

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> taylor(z^5*exp(z)/sin(z)^5,z=0,10);
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$$1+z+\frac{4}{3}z^2+z^3+\frac{5}{6}z^4+\frac{47}{90}z^5+\frac{326}{945}z^6+\frac{361}{1890}z^7+\frac{2477}{22680}z^8+\frac{179}{3240}z^9+O(z^{10})$$

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> series(exp(z)/sin(z)^5,z=0,10);
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$$z^{-5}+z^{-4}+\frac{4}{3}z^{-3}+z^{-2}+\frac{5}{6}z^{-1}+\frac{47}{90}+\frac{326}{945}z+\frac{361}{1890}z^2+\frac{2477}{22680}z^3+\frac{179}{3240}z^4+O(z^5)$$

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>
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Residue thm examples :

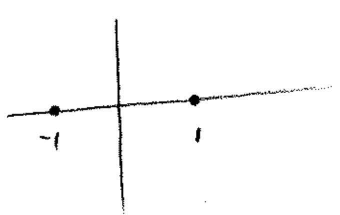
$$\int_\gamma \frac{z^2}{(z-1)^2(z+1)}\,dz$$

where

(1) $\gamma(t) = 1 + e^{it}$ $\quad 0 \le t \le 4\pi$

(2) $\gamma(t) = 2e^{it}$ $\quad 0 \le t \le 2\pi$

(3) $\gamma(t) = i + e^{it}$ $\quad 0 \le t \le 2\pi$



The residue sledgehammer for poles of arbitrary order (nice to know it exists, but healthy to use the *method* rather than the result).
(see pages 249-250)

Let $f(z) = \frac{g(z)}{h(z)}$ have a pole of order k at z_0,
with $g(z_0) \neq 0$
$h(z_0)$ a zero of order k.

residue!

analytic at z_0

Thus near z_0,

$$\frac{g(z)}{h(z)} = \left[\frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0} + \rho(z) \right]$$

$$(z-z_0)^k \underbrace{g(z)}_{\sum_{n=0}^{\infty} b_n (z-z_0)^n} = \underbrace{h(z)}_{\sum_{n=k}^{\infty} c_n (z-z_0)^k} \left[a_{-k} + a_{-k+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{k-1} + \rho(z)(z-z_0)^k \right]$$

powers of $(z-z_0)$:	eqtn:
k	$b_0 = c_k a_{-k}$
$k+1$	$b_1 = c_{k+1} a_{-k} + c_k a_{-k+1}$
$k+2$	$b_2 = c_{k+2} a_{-k} + c_{k+1} a_{-k+1} + c_k a_{-k+2}$
$\vdots$	
$2k-1 = (k+(k-1))$	$b_{k-1} = c_{2k-1} a_{-k} + \cdots + c_k a_{-1}$

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \end{bmatrix} = \begin{bmatrix} c_k & & 0 \\ c_{k+1} & c_k & \\ \vdots & & \ddots \\ c_{2k-1} & & c_k \end{bmatrix} \begin{bmatrix} a_{-k} \\ a_{-k+1} \\ \vdots \\ a_{-1} \end{bmatrix}$$

Cramer's rule:

$$\boxed{a_{-1} = \frac{1}{(c_k)^k} \begin{vmatrix} c_k & 0 & b_0 \\ c_{k+1} & c_k & b_1 \\ \vdots & \ddots & \vdots \\ c_{2k-1} & & b_{k-1} \end{vmatrix}}$$

(you could check the Fri page 4 examples with this)