Math 4200
Fri 10/28

Hw for Fri 11/4
4.1 1de, 3, 5, 7ab, 9
4.2 2, 3, 4, 6, 8, 13, 15

- Finish isolated singularities table.

- Multiplying Laurent series term by term – is it justified? e.g. #6 §3.3
  We justified this procedure for Taylor series by computing $\frac{(fg)^{(n)}(z_0)}{n!}$
  won't work here!

  4.1 #12 outlines one way to justify this (not assigned)

  Here's another (I think!)

Theorem: Let $f(z), g(z)$ have Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

in $D(z_0,R) \setminus \overline{D(z_0,r)}$ $\qquad g(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$

Then the Laurent series for $f(z)g(z)$ is $\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$

where $c_n = \lim_{N\to\infty} \sum_{j=-N}^{N} a_j b_{n-j}$

proof. Let γ be a C' closed curve in the annulus with $I(\gamma; z_0) = 1$, then

$$* \qquad c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)g(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta$$

Let $f_N(z) = \sum_{j=-N}^{N} a_j (z-z_0)^j$, $\quad g_N(z) = \sum_{k=-N+n}^{N+n} b_k (z-z_0)^k$

$f_N \to f$, $g_N \to g$, uniformly on compact subannuli, so uniformly on γ

$\Rightarrow f_N g_N \to fg$ uniformly on γ

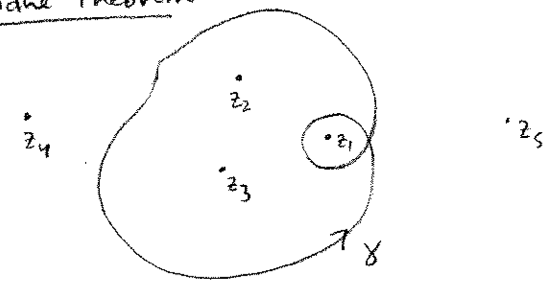
$* \Rightarrow c_n(f_N g_N) \to c_n(fg)$ uniformly

$\qquad \parallel$

$\sum_{j=-N}^{N} a_j b_{n-j}$, by explicit multiplication of the finite sums for f_N & g_N!! ∎

(This proof would've worked for Taylor series too.)

Residue Theorem

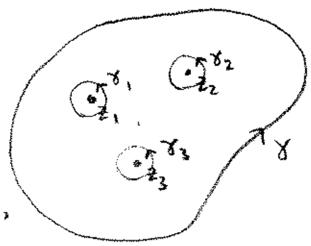


A open, γ closed, homotopic to a point in A

$f : A \setminus \{z_1, \ldots z_n\} \longrightarrow \mathbb{C}$ analytic

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)\, I(\gamma, z_j)$$

we'll give full-blown proof later,
but special cases easy!



If γ bounds a region, D

$$\oint_\gamma f(z)\,dz = \sum_{z_j \text{ inside region}} \int_{\gamma_j} f(z)\,dz$$

$$\int_\gamma - \sum_j \int_{\gamma_j}$$

$$\parallel$$

e.g. $\int_{\partial D} f(z)\,dz = 0$ Green's.

or add bridging contours from γ to each γ_j to make a domain w/o holes, etc.

We may choose each γ_j in a region near enough z_j, so that the Laurent expansion about z_j converges there:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n \quad \text{in } D(z_j, r_j) \setminus \{z_j\}$$

$$\Rightarrow \int_{\gamma_j} f(z)\,dz = \int_{\gamma_j} \sum = \sum \int_{\gamma_j} = \sum_{n=-\infty}^{\infty} a_n \int_{\gamma_j} (z - z_j)^n\,dz = 2\pi i\, a_{-1}$$

(all other terms $= 0$!)

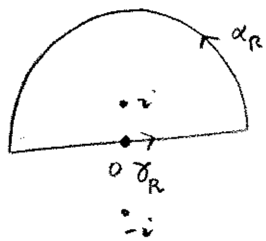
$$= 2\pi i \operatorname{Res}(f, z_j)$$

$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{z_j \text{ inside } \gamma} \int_{\gamma_j} f(z)\,dz$$

$$= 2\pi i \sum_{z_j \text{ inside } \gamma} \operatorname{Res}(f, z_j) \qquad \blacksquare$$

example(s).
We may revisit some old friends:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx$$



$$\int_{-R}^{R} \frac{1}{1+x^2}\,dx = \int_{\gamma_R} f(z)\,dz \qquad f(z) = \frac{1}{1+z^2}$$

$$= \frac{1}{(z-i)(z+i)} \quad \text{analytic near } i$$

$$= \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$\gamma := \gamma_R + \alpha_R \qquad \mathrm{Res}(f, i) = \frac{1}{2i}$$

$$\int_\gamma f(z)\,dz = 2\pi i \;\mathrm{Res}(f; i) = 2\pi i \,\frac{1}{2i} = \pi$$

(independent of R)

$$\left| \int_{\alpha_R} f(z)\,dz \right| \le \int_{\alpha_R} \frac{1}{|z|^2 - 1}\,|dz| = \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty$$

So $\displaystyle\int_{\gamma_R} f(z)\,dz = \int_\gamma f(z)\,dz - \int_{\alpha_R} f(z)\,dz$

$\to \pi$ as $R \to \infty$.

$\to \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

So $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$.

So, we need good ways to compute residues.
There is a very detailed chart on page 250 (§4.1)
We don't need to verify all entries there, but let's try (4), (6) to get the flavor:

$$(4): \quad f(z) = \frac{g(z)}{h(z)} \qquad \begin{matrix} g(z_0) \ne 0 \\ h(z_0) = 0,\ h'(z_0) \ne 0 \end{matrix} \implies \mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

(simple pole for f)

pf

$$g(z) = \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}{b_1(z-z_0) + b_2(z-z_0)^2 + \cdots} \qquad \begin{matrix} a_0 \ne 0 \\ b_1 \ne 0 \end{matrix}$$

example $f(z) = \dfrac{1}{1+z^2} \begin{matrix} \leftarrow g \\ \leftarrow h \end{matrix}$

$\mathrm{Res}(f, i) = \dfrac{1}{2i}$ ✓

$$= \frac{1}{(z-z_0)} \Bigg[\underbrace{\frac{a_0 + a_1(z-z_0) + \cdots}{b_1 + b_2(z-z_0) + \cdots}}_{\text{analytic at } z_0,\ \text{value there} = \frac{a_0}{b_1}} \Bigg]$$

$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad c_0 = \frac{a_0}{b_1}$$

$$\Rightarrow g(z) = \overset{\mathrm{Res}(f, z_0)\ \blacksquare}{\overset{\downarrow}{\frac{a_0}{b_1}}} \frac{1}{z-z_0} + \sum_{n=1}^{\infty} c_n (z-z_0)^{n-1}$$

(6): double pole,

$$f(z) = \frac{g(z)}{h(z)} \qquad g(z_0) \neq 0, \quad h(z_0) = h'(z_0) = 0. \qquad \Rightarrow \operatorname{Res}(f, z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)\, h'''(z_0)}{h''(z_0)^2}$$

looks imposing!

pf assume $z_0 = 0$ (can reduce to this case by making the substitution $w = z - z_0$)

$$f(z) = \frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{n=2}^{\infty} b_n z^n} \qquad a_0 \neq 0$$

$$= \frac{1}{z^2}\left(\frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{n=2}^{\infty} b_n z^{n-2}}\right)$$

↑ analytic near 0

$$= c_0 + c_1 z + c_2 z^2 + \cdots$$

$$\operatorname{Res}(f, 0) = c_1$$

multiply thru:

$$(a_0 + a_1 z + a_2 z^2 + \cdots) = (\overset{a_0/b_2}{\cancel{c_0}} + c_1 z + c_2 z^2 + \cdots)(b_2 + b_3 z + b_4 z^2 + \cdots)$$

$$a_0 = c_0 b_2, \quad c_0 = \frac{a_0}{b_2}$$

$$a_1 = c_1 b_2 + b_3 c_0$$
$$= c_1 b_2 + \frac{b_3 a_0}{b_2}$$

$$c_1 = \frac{a_1}{b_2} - \frac{b_3 a_0}{b_2^2}$$
$$= \frac{g'(0)}{\frac{1}{2}h''(0)} - \frac{\frac{1}{6}h'''(0)\, g(0)}{\frac{1}{4}h''(0)^2} \quad \checkmark$$

<u>example</u> $\quad f(z) = \dfrac{e^z}{z \sin z} \qquad g(0) = g'(0) = 1$

$$z \sin z = z\left(z - \frac{z^3}{3!} + \cdots\right)$$
$$= z^2 - \frac{z^4}{3!} + \cdots$$

$$h''(0) = 2, \quad h'''(0) = 0 \quad \Rightarrow \operatorname{Res}(f, 0) = \frac{2 \cdot 1}{2} - 0 = 1.$$

or

$$\frac{1 + z + \frac{z^2}{2!} + \cdots}{z^2 - \frac{z^4}{3!} + \cdots} = \left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots\right)$$

$$1 + z + \frac{z^2}{2} + \cdots = \left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots\right)\left(z^2 - \frac{z^4}{3!} + \cdots\right)$$

$$1 = a_{-2}$$
$$1 = a_{-1} + 0 \quad \checkmark$$