93.3: Classification of isolated singularities

If \( f \) analytic in \( D(z_0, r) \setminus \{z_0\} \),
then the singularity at \( z_0 \) is called isolated.

<table>
<thead>
<tr>
<th>Name (type) of isolated singularity at ( z_0 )</th>
<th>Laurent expansion characterization</th>
<th>Characterization in terms of ( \lim_{z \to z_0} f(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>removable (because you can extend ( f ) to be analytic at ( z_0 ))</td>
<td>( f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n )</td>
<td>( \lim_{z \to z_0} f(z) \neq \infty ) as a finite number</td>
</tr>
<tr>
<td>pole (means North)</td>
<td>( f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n ) ( \text{N} \in \mathbb{N} ) ( a_N \neq 0 )</td>
<td>( \lim_{z \to z_0} f(z) = \infty ), ( z_0 )</td>
</tr>
<tr>
<td>simple pole</td>
<td>( f(z) = \sum_{n=-1}^{\infty} a_n (z-z_0)^n ) ( a_{-1} \neq 0 )</td>
<td>( \exists \text{N} \in \mathbb{N} ) s.t. ( g(z) := f(z)(z-z_0)^N ) has a removable singularity at ( z_0 ), and ( g(z) \neq 0 )</td>
</tr>
</tbody>
</table>
| essential singularity | \( f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \) and \( \exists n \to -\infty \) with \( a_n \neq 0 \) | \( f(D(z_0, \epsilon) \setminus \{z_0\}) = C \) ! \([\text{In fact, a hand result, called Picard's Theorem, says} f(D(z_0, \epsilon) \setminus \{z_0\}) \text{ contains all of } C \text{ except for at most two points!} ]\)

\[ \text{Example: } e^{1/z}, \ z_0 = 0. \]

Notice that the 3 possibilities for Laurent expansions partition the set of all possibilities (i.e., they're mutually exclusive, and each Laurent expansion satisfies one of the three possibilities).

We must show that the limit characterizations correspond to the Laurent series partitioning.
removable

singularity

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{(Laurent characterization)} \]

in \( D(z_0, r) \setminus \{ z_0 \} \).

Then, since this power series also converges at \( z_0 \), it defines an analytic function in \( D(z_0, r) \).

\( \Rightarrow \) 1. \( \lim_{z \to z_0} f(z) = f(z_0) = a_0 \) exists (since analytic \( \Rightarrow \) continuous).

\( \Rightarrow \) 2. \( f \) bounded near \( z_0 \); in fact for \( M = \max |f(z)| \), with \( 0 < r - z_0 < \delta \) \( f(z) \leq M \).

\( \Rightarrow \) 3. \( \lim_{z \to z_0} |f(z) - f(z_0)| = \lim_{z \to z_0} |M - z| = 0 \).

So \( \lim_{z \to z_0} f(z) = f(z_0) = 0 \).

The circle is completed if we show \( 3 \Rightarrow \) Laurent characterization.

Recall, if \( \delta(t) = \rho e^{it} \), \( 0 \leq t \leq 2\pi \).

Then

\[ a_n = \frac{1}{2\pi i} \int_{\delta(t)} f(z) \frac{dz}{(z-z_0)^{n+1}} \quad \forall n, \text{ (See Monday notes)} \]

Let \( n \leq -1 \). We show \( a_n = 0 \).

\[ |a_n| = \frac{1}{2\pi} \int_{\delta(t)} |f(z)| \frac{dz}{|z-z_0|^{n+1}} \leq \frac{\max |f(z)|}{r^{n+1}} \]

\[ \leq \frac{1}{2\pi} \left[ \max |f(z)| \right] \frac{2\pi \rho}{\rho^{n+1}} \]

\[ = \left[ \max |f(z)| e \right] \frac{1}{r^{n+1}} \]

As \( \rho \to 0 \) if \( n < -1 \).

\[ 0 \quad \text{else} \to 0 \]

Thus \( \nabla \).

To complete the implication circle, must show \( 1 \Rightarrow \) Laurent characterization.

pole

Laurent characterization

\( \Rightarrow \) \( g(z) = (z - z_0)^N f(z) \)

satisfies

\[ g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+N} \]

\[ = \sum_{j=0}^{\infty} a_{j-N} (z - z_0)^j \]

\( \nu = \sum_{j=0}^{\infty} b_j (z - z_0)^j \), \( \text{with } b_0 \neq 0 \).

\( \Rightarrow \) 2. \( g(z) \) has a removable singularity at \( z_0 \).

But \( 2 \Rightarrow \) \( f(z) = \frac{g(z)}{(e-z_0)^N} \)

and

\[ \lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} |g(z)| \] \( \lim_{z \to z_0} \frac{1}{(z-z_0)^N} = +\infty \). \( \nabla \)
digress into meaning of “pole”, which goes back to cartographer/surveyor Gauss.

A more compact setting (C) for complex analysis is the Riemann sphere, (see 41.4)

$$S^2_R := \mathbb{C} \cup \{\infty\}$$

$$\lim_{n \to 0} z_n = \infty \quad \text{iff} \quad \lim_{n \to 0} \frac{1}{z_n} = 0.$$ 

$S^2_R$ is topologically a sphere because:

\[ \begin{array}{c}
\text{St}^{-1} \quad \text{stands for}
\text{the inverse of stereographic projection from the}
\text{unit sphere in } \mathbb{R}^3 \\
\text{to the complex plane}
\end{array} \]

under the inverse to stereographic projection from the north pole, C corresponds to all of the unit sphere $S^2$, except N, and $\infty$ corresponds to N.

In fact, $\lim_{z \to \infty} f(z) = \infty$ means $\text{St}^{-1}(f(z)) \to N$, the north pole.

If you wish to study analytic maps with domain/range in $S^2_R$, you can get by with an atlas of 2 coord charts:

\[ \begin{array}{c}
(C \cup \{0\}) \setminus \{\infty\}
\end{array} \]

\[ \begin{array}{c}
\{w \in \mathbb{C} \mid \frac{1}{w} \}
\end{array} \]

\[ \begin{array}{c}
\frac{1}{w} \quad \text{describes things near } 0
\end{array} \]

\[ \begin{array}{c}
\text{also}
\end{array} \]

\[ \begin{array}{c}
\text{describes things near } \infty
\end{array} \]
This is almost the same as studying $S^2$ with 2 charts, corresponding to stereographic projection from the north & south poles.

\[ N \quad P \quad S \quad W = S'_N(P) \]

in this case:
\[ \text{rectangular cross section } \mathbb{R}^3 \]

by similar $\triangle$'s,
\[ \frac{121}{1} = \frac{1}{1 W} \Rightarrow W = \frac{1}{121} \]
\[ \Rightarrow W = \frac{2}{121} = \frac{1}{61} \text{ almost } \quad W = \frac{1}{61} \quad \text{but orientation reversing.} \]

back to pole considerations:
(1) \Rightarrow Lament.

\[ \lim_{z \to \infty} f(z) = \infty. \quad \text{Define } k(z) := \frac{1}{f(z)} \quad \text{(in other words we're using our W-chart to study } S^2) \]

\[ \lim_{z \to z_0} k(z) = 0 \]
\[ \Rightarrow k \text{ has removable singularity at } z_0 \quad \& \quad k(z_0) = 0 \]
\[ \Rightarrow k(z) = \sum_{n=N}^{\infty} a_n (z-z_0)^n = (z-z_0)^N h(z), \quad h(z_0) \neq 0 \quad \text{analytic} \]
\[ \Rightarrow f(z) = \frac{1}{k(z)} = \frac{1}{(z-z_0)^N} \left[ \frac{1}{h(z)} \right] \]
\[ = \frac{1}{(z-z_0)^N} \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{analytic near } z_0 \quad \text{since } h(z_0) \neq 0 \]
Since the three Laurent series characterizations are mutually exclusive & exhaustive, since we already showed the correspondence between Laurent & limit descriptions for "removable" and "pole", and since the limit characterizations are mutually exclusive, we will be done if we can show the limit conditions are exhaustive! (By pure logic!) (since then both limit & laurent characterizations of essential singularities can be rephrased as not removable or pole!)

So, assume it is not true that

\[ \forall \delta \in \mathbb{C}, \quad f(D(0, \delta) \setminus \{0\}) = \emptyset \]

then \[ \exists \delta \in \mathbb{C}, \, \delta > 0 \quad \text{s.t.} \quad 1/(f(z)-w_0) > \varepsilon \quad \forall z \in D(z_0, \delta) \setminus \{z_0\} \]

Define \[ k(z) = \frac{1}{f(z) - w_0} \]

for \[ 0 < |z - z_0| < \delta \]

\[ k \text{ has a removable singularity at } z_0 \]

\[ k(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n \quad N \geq 0, \quad a_N \neq 0 \]

\[ k(z) = (z - z_0)^N h(z) \quad h(z_0) \neq 0 \]

\[ \frac{1}{f(z) - w_0} = (z - z_0)^N h(z) \]

\[ f(z) = w_0 + \frac{1}{(z - z_0)^N \frac{1}{h(z)}} \]

analytic near \( z_0 \) since \( h(z_0) \neq 0 \)

\( N = 0 \rightarrow \text{removable singularity} \)

\( N > 1 \rightarrow \text{pole of order } N \)
Def: $f$ is meromorphic in $A$ if it has at most isolated singularities, none of which are essential

[so $f$ can be considered an analytic map to $S^2_{\mathbb{C}}$]

Def: a zero of order $-n$ $(n \in \mathbb{N})$ is a pole of order $n$.

Then: Let $f$ have a zero of order $n$ at $z_0$, $g$ has a zero of order $m$ at $z_0$.

Then $\frac{f}{g}$ has a zero of order $n-m$.

pf: $f(z) = (z-z_0)^nf_1(z)$
$g(z) = (z-z_0)^mg_1(z)$

$\Rightarrow f(z) = (z-z_0)^{n-m} g(z) h(z)$

where $f_1(z_0) \neq 0$ and $g_1(z_0) \neq 0$.

Then $h(z) \neq 0$, $h(z) = \frac{f_1(z)}{g_1(z)}$.

Example: What is the order of the zero of

$$f(z) = \frac{\sin(z^4)}{(e^{2z}-1)^2}$$

at $z = 0$?