Math 4200
Fri Oct 21

Hw for Fri 10/28

(these may get postponed) →

3.3 1ab, 2, 4, 6, 8, 9, 13, 15, 17, 18, 19, 20
4.1 1de, 3, 5, 7ab, 9, 12

- For §3.1 I should have explicitly highlighted

Theorem 3.1.7 (Weierstrass M-test)

Let $\{g_n\}$, $g_n : A \to \mathbb{C}$
suppose $\exists\ M_n \geq 0$ s.t.

(i) $|g_n(z)| \leq M_n \quad \forall z \in A, \forall n$

(ii) $\sum_{n=1}^{\infty} M_n$ converges (equivalent to "is bounded above")

Then $\sum_{n=1}^{\infty} g_n(z)$ converges absolutely & uniformly on A.

(we've been using this a lot in our radius of convergence discussions, but it gives you a framework to do the §3.1 HW problems).

pf of thm
$$S_N(z) - S_{\tilde{N}}(z) = \sum_{n=\tilde{N}+1}^{N} g_n(z) \qquad (\tilde{N} < N)$$
$$|S_N(z) - S_{\tilde{N}}(z)| \leq \sum_{n=\tilde{N}+1}^{N} |g_n(z)| \leq \sum_{n=\tilde{N}+1}^{N} M_n \to 0 \text{ as } \tilde{N}, N \to \infty$$

so $\{S_N(z)\}$ is uniformly Cauchy, as is the series $\sum |g_n(z)|$ ▣

- Finish cloning theorems from Wed.

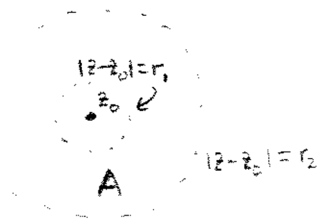
Note: Theorem 1 also has a quick proof using the fact that the only non-empty open & closed subset of a connected set is the entire set.

Hint: consider $S = \{ z_1 \in A$ s.t. the power series for f at z_1 is identically zero $\}$

why is S open?
why is S closed?

- Do second cloning theorem.

Laurent Series : analytic fns defined on annuli can be written as power series with $n \in \mathbb{Z}$ (positive & negative powers). & vise versa



Theorem Let $0 \le r_1 < r_2 \le \infty$.
$z_0 \in \mathbb{C}$
$A := \{ z \in \mathbb{C} \text{ s.t. } r_1 < |z-z_0| < r_2 \}$
$f : A \to \mathbb{C}$ analytic

(1) Then $\ast$ $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$;

where each series converges $\forall z \in A$.

(2) Conversely, if $f(z)$ is a sum of two convergent series $\ast$ in A, then the convergence is uniform in each subannulus $B_{\rho_1 \rho_2} = \{ z \in \mathbb{C} \mid r_1 < \rho_1 \le |z-z_0| \le \rho_2 < r_2 \}$,

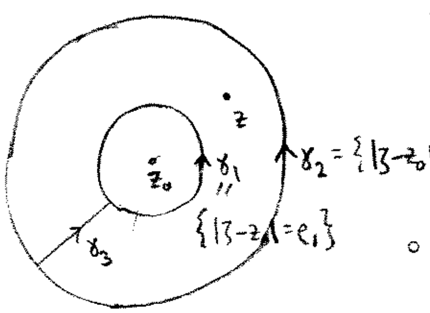
so f is analytic.

(3) The coefficients $\{a_n\}$, $\{b_m\}$ are uniquely determined by the analytic fcn f.

Existence of Laurent series for f

proof Let $z \in A$
pick $\underset{r_1}{\overset{\vee}{\rho_1}} < |z-z_0| < \underset{r_2}{\overset{\wedge}{\rho_2}}$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$



(because $\gamma = \gamma_2 + \gamma_3 - \gamma_1 - \gamma_3$ is homotopic to a curve with index 1 in a simply connected subdomain of f, so

$$f(z)\, I(\gamma; z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \;)$$

C.I.F.

on γ_2, $|\zeta - z_0| > |z - z_0|$
proceed as in Taylor series (exactly).

$$\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} d\zeta$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0} \left(\underbrace{\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}}_{\sum_{n=0}^{\infty} \left(\frac{z-z_0}{\zeta - z_0}\right)^n} \right) d\zeta$$

unif conv wrt $\zeta \in \gamma_2$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta}_{a_n}$$

on γ_1, $|z - z_0| > |\zeta - z_0|$, so

$$-\frac{1}{2\pi i} \int_{\gamma_1} - \frac{f(\zeta)}{(z - z_0) - (\zeta - z_0)} d\zeta$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{z - z_0} \left(\underbrace{\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}}_{\sum_{k=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^k} \right) d\zeta$$

$$= \sum_{n=1}^{\infty} \frac{1}{(z - z_0)^n} \underbrace{\frac{1}{2\pi i} \int_{\gamma_1} f(\zeta)(\zeta - z_0)^{n-1} d\zeta}_{b_n}$$

$\Longrightarrow \ast$ holds.

so, $f: A \to \mathbb{C}$ analytic

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} b_m/(z-z_0)^m$$

with $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta$, $b_m = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta-z_0)^{m-1}\,d\zeta$

γ <u>any</u> curve homotopic to γ_1 (or γ_2) in A

<u>Now for converse</u>:

Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty} b_m/(z-z_0)^m$ converge in A

$= S_1(z) + S_2(z)$

Then S_1 converges uniformly absolutely $\forall\ |z-z_0| \le \rho_2 < r_2$

— We did this when we were doing power series!

And $S_2(z)$ converges uniformly absolutely $\forall\ |z-z_0| \geq \rho_1 > r_1$, HW §3.3 #15.

$\Rightarrow$ each series converges uniformly absolutely in $B_{\rho_1\rho_2}$

$\Rightarrow$ for $a_{-n} := b_n$,

$\lim_{N\to\infty} \sum_{n=-N}^{N} a_n(z-z_0)^n = f(z)$ and limit is uniform on $B_{\rho_1\rho_2}$

$\Rightarrow f$ analytic in $\operatorname{int}(B_{\rho_1\rho_2})$

$\Rightarrow f$ analytic in A

<u>Uniqueness</u>: let f and its Laurent expansion be given. Amalgamate each series to

Write $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$

Let $\gamma: I \to A$ a closed curve with $I(\gamma; z_0) = 1$.

$$\int_\gamma f(\zeta)\,d\zeta = \int_\gamma \sum_{-\infty}^{\infty} (\ \ )\,d\zeta = \sum_{n=-\infty}^{\infty} \int_\gamma a_n(\zeta-z_0)^n\,d\zeta = \begin{cases} 0 & n \neq -1 \quad \text{FTC} \\ a_{-1}\,2\pi i & n=-1, \text{ since } I(\gamma; z_0)=1 \end{cases}$$

meaning $\lim_{N\to\infty} \sum_{-N}^{N}$

so $\int_\gamma f(\zeta)\,d\zeta = (a_{-1})(2\pi i)$

For this reason a_{-1} is called "<u>the residue of f at z_0</u>", and will be the focus of chptr 4.

But now,

$$\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta = \sum_{n=-\infty}^{\infty} \int_\gamma a_n \frac{(\zeta-z_0)^n}{(\zeta-z_0)^{k+1}}\,d\zeta = \begin{cases} 0 & n \neq k \\ a_k\,2\pi i & n = k \end{cases}$$

$$\boxed{a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta \quad \forall\ k \in \mathbb{Z}}$$

Shows a_k's are uniquely determined by f!

examples

$f(z) = e^{\frac{1}{z}}$

$z_0 = 0$

$0 < |z| < \infty$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z^2}\right) + \cdots + \sum_{j=0}^{\infty} \frac{1}{j!\, z^j}$$

$f(z) = \frac{1}{1-z^2}$

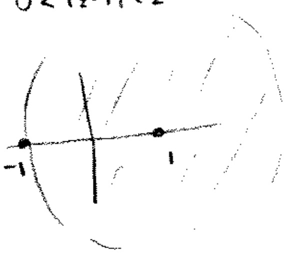
$z_0 = 0$, $0 \le |z| < 1$ $\Big\}$ f analytic; $\quad f(z) = 1 + z^2 + z^4 + \cdots = \sum_{j=0}^{\infty} z^{2j}$

$z_0 = 0$, $|z| > 1$

$$f(z) = \frac{1}{z^2}\left(\frac{1}{\frac{1}{z^2}-1}\right) = -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) = -\frac{1}{z^2}\left[1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right] = -\frac{1}{z^2}\sum_{j=0}^{\infty} z^{-2j}$$

$z_0 = 1$

$0 < |z-1| < 2$



$$\frac{1}{1-z^2} = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right) = -\frac{1}{2}\,\frac{1}{z-1} + \underbrace{\frac{1}{2}\,\frac{1}{(z-1)-(-1-1)}}_{\underbrace{\frac{1}{2}\,\frac{1}{2+(z-1)}}_{\underbrace{\frac{1}{4}\,\frac{1}{1+\frac{z-1}{2}}}_{\sum_{j=0}^{\infty}\left(\frac{z-1}{2}\right)^j}}}$$

$0 < |z-1| < 2 \Rightarrow$

$$\frac{1}{1-z^2} = -\frac{1}{2}\,\frac{1}{(z-1)} + \frac{1}{4}\sum_{j=0}^{\infty} \frac{(z-1)^j}{2^j}$$

Could you do it for $|z-1| > 2$?

$\frac{e^z}{\sin z} = f(z)$

$z_0 = 0.$

$0 < |z| < \pi$

note $\frac{z e^z}{\sin z} = \frac{e^z}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots}$ is analytic (after defining its value at 0 to be 1) for $0 \le z < \pi$

$$= \sum_{j=0}^{\infty} a_j z^j$$

$$\Rightarrow \frac{e^z}{\sin z} = \sum_{j=0}^{\infty} a_j z^{j-1} = \sum_{j=-1}^{\infty} b_j z^j \quad \text{(reindex)}$$

$$\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = \left(\frac{\overset{1}{b_{-1}}}{z} + \overset{1}{b_0} + \underset{1/3}{b_1 z} + b_2 z^2 + b_3 z^3 + \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

z^0: $\quad 1 = b_{-1}$

z^1: $\quad 1 = b_0$

z^2: $\quad \frac{1}{2} = -\frac{1}{6} + b_1$; $\; b_1 = \frac{1}{3}$

z^3: $\quad \frac{1}{6} = -\frac{1}{6} + b_2$; $\; b_2 = \frac{1}{3}$ $\quad$ etc.