Math 4200
Wed 18 Oct.

- Finish page 4 Monday,
  about multiplying power series term by term

example (from differential eqtns)

[There is a general thm which says if you have an n^{th} order linear DE with analytic coefficients in a nbhd of z_0, with $a_n(z) \neq 0$ there, then this method will provide you with an analytic soltn to IVP in a nbhd of z_0. You use this thm to get series expansions for things like Bessel's fcns. ~ Using $\mathbb{C}$-analysis for real applications].

Solve

$$\begin{cases} f''(z) + f(z) = 0 \\ f(0) = 1 \\ f'(0) = 0 \end{cases}$$

Let $f(z) = \sum a_n z^n$

$f(0) = 1 \Rightarrow a_0 = 1$
$f'(0) = 0 \Rightarrow a_1 = 0.$

$$f'' + f = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$k = n-2 \qquad \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} z^k + \sum_{k=0}^{\infty} a_k z^k \equiv 0.$$

$(k+2)(k+1) a_{k+2} = -a_k$ recursion relation.

$$a_{k+2} = \frac{-1}{(k+2)(k+1)} a_k$$

$a_0 = 1 \Rightarrow a_2 = -\frac{1}{2} \Rightarrow a_4 = \frac{1}{4!}, \ a_6 = -\frac{1}{6!}, \ldots \quad a_{2k} = (-1)^k \frac{1}{(2k)!}$
$a_1 = 0 \Rightarrow a_3 = 0 \Rightarrow a_k = 0$ k odd.

$$f(z) = \sum_k (-1)^k \frac{1}{(2k)!} z^{2k} = \cos z \ \checkmark$$

Another application of the fact that analytic fcns have local (unique) power series is

Uniqueness of analytic continuation (uniqueness of clones :>)

Theorem 1: Let A be open, connected, $f,g: A \to \mathbb{C}$ analytic

Let B open, $B \neq \emptyset$, $B \subset A$.

Let $f \equiv g$ on B.

Then $f \equiv g$ on A

proof: Let $h(z) = f(z) - g(z)$

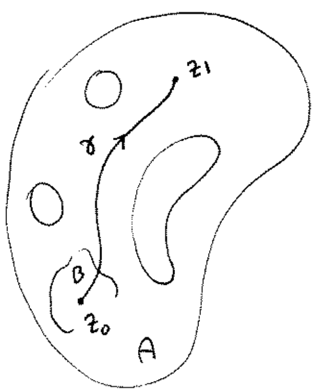
We will show $h \equiv 0$ on A.

Pick $z_0 \in B$. Let $z_1 \in A$.

Let $\gamma : [0,1] \to A$ be continuous

$\gamma(0) = z_0$

$\gamma(1) = z_1$



since $\gamma([0,1])$ is compact,

distance $(\gamma([0,1]), \mathbb{C} \setminus A) \geq \rho > 0$

i.e. $\exists \rho > 0$ s.t. $D(\gamma(t), \rho) \subset A \quad \forall 0 \leq t \leq 1$.

Let

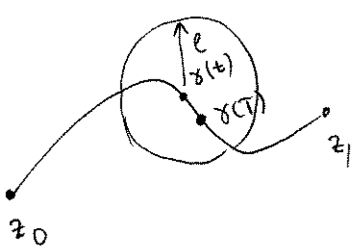
$$T := \sup \left\{ t \in [0,1] \text{ s.t. power series for } h(z), \text{ centered at } \gamma(s), \text{ is identically zero}, \forall 0 \leq s \leq t \right\}$$

Note $T > 0$ since $h \equiv 0$ in B

and $z_0 \in B \Rightarrow \gamma([0,\delta)) \subset B$ some $\delta > 0$.

Pick $t < T$, with $|\gamma(t) - \gamma(T)| < \rho/2$

Since $t < T$, $\quad h(z) = \sum_{j=0}^{\infty} 0 \cdot (z - \gamma(t))^j$



$\forall z \in D(\gamma(t), \rho)$ (By Cauchy)

$\cup$

$D(\gamma(T), \rho/2)$ $\quad \Delta$ ineq

$\Rightarrow h^{(k)}(\gamma(T)) = 0 \quad \forall k$

$\Rightarrow h(z) = \sum_{j=0}^{\infty} 0 (z - \gamma(T))^j \equiv 0 \quad \forall z \in D(\gamma(T), \rho)$

$\Rightarrow T = 1$, (since $T < 1$ would imply $\exists\, t > T$ where ...) $\blacksquare$

Example Determine the radius of convergence for the power series for $\tan z = \frac{\sin z}{\cos z}$, at $z = 0$ without actually computing the power series.

$\cos z = 0$

$\frac{1}{2}(e^{iz} + e^{-iz}) = 0$

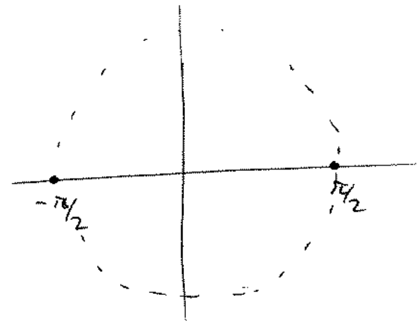
$e^{2iz} + 1 = 0$

$e^{2iz} = -1$

$2iz = \log(-1)$

$= 0 + i(\pi + 2\pi n)$

$z = \frac{\pi}{2} + n\pi$



$R \geq \pi/2$ by Cauchy.

If $R > \pi/2$ then let $g(z)$ be the analytic fcn from $\tan z$ which the power series produces.

Note $g(z) = \tan z$ on $D(0, \pi/2)$ by construction.

$\Rightarrow \lim_{z \to \pi/2} |g(z)| = +\infty$

contradicts $R > \pi/2$!

$\boxed{\therefore R = \pi/2}$

Theorem 2 (Supercloning)

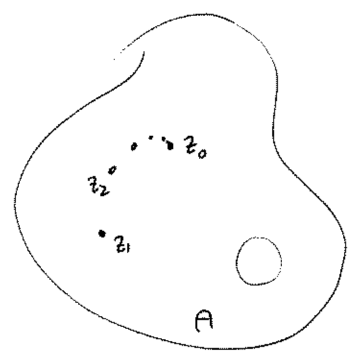
Let A open, connected

$f, g : A \to \mathbb{C}$ analytic

Suppose $\exists \{z_n\} \subset A$, $\{z_n\} \to z_0 \in A$, $z_n \neq z_0$ $\forall n \geq 1$.

Suppose $f(z_n) = g(z_n)$ $\forall n$

Then $f \equiv g$ in A!



proof: Let $h(z) = f(z) - g(z)$.

Show $h \equiv 0$ in A.

Expand $h(z)$ in power series @ z_0: $h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

Case I $a_n = 0$ $\forall n \Rightarrow h \equiv 0$ in open disk about $z_0 \Rightarrow h \equiv 0$ by Theorem 1.

Case II $\exists N$ s.t. $a_N \neq 0$, $a_n = 0$ $\forall n < N$.

$$\Rightarrow h(z) = a_N (z - z_0)^N + \sum_{j=N+1}^{\infty} a_j (z - z_0)^j$$

$$\Rightarrow \frac{h(z)}{(z - z_0)^N} = a_N + \underbrace{\sum_{j=N+1}^{\infty} a_j (z - z_0)^{j-N}}_{\text{analytic, so cont at } z_0}$$

$$\Rightarrow \lim_{z \to z_0} \frac{h(z)}{(z - z_0)^N} = a_N.$$

This would be violated if $\exists \{z_n\} \to z_0$ ($z_n \neq z_0$) with $h(z_n) = 0$!

So Case II doesn't occur! ∎

<u>Corollary</u> : If $f : A \to \mathbb{C}$ is analytic, $f \not\equiv 0$,

then $f(z_0) = 0 \Rightarrow \exists\, \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow f(z) \neq 0$.

i.e. zeroes are "isolated"

<u>proof</u> : If z_0 is not an isolated zero $\exists\, \{z_n\} \subset A$, $\{z_n\} \to z_0$, $z_n \neq z_0\ \forall z$

$f(z_n) = 0$

In which case $\underset{\substack{\uparrow \\ \text{"}g\text{"}}}{f \equiv 0}$ in A, by Thm 2 (or by its proof.)

∎