Math 4200
Monday Oct 17.

Recall on Friday we showed
$\overline{D(z_0,r)} \subset A_{\text{open}}$
$f: A \to \mathbb{C}$ analytic
Then $|z-z_0|<r$
$$\Rightarrow f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(z_0)\,(z-z_0)^j = \sum_{j=0}^{\infty} (z-z_0)^j \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta$$

notice convergence is ~~a~~ uniformly absolute on $\overline{D(z_0,\rho)}$ with $\rho<r$.

Do you recall how we derived this?

So every analytic fcn has a Taylor series expansion at z_0, and it will converge on any disk of analyticity.

We are studying the converse.
Finish page ③ Friday, to show for given $\{a_j\} \subset \mathbb{C}$
if $g(z) := \sum_{j=0}^{\infty} a_j (z-z_0)^j$ (for z at which series converges)

then for
$R := \sup\{|z-z_0| \text{ s.t. } g(z) \text{ converges}\}$ (R = radius of convergence)
then $|z-z_0| < R \Rightarrow$ abs conv
$|z-z_0| > R \Rightarrow$ div.

<u>Theorem 1</u> Let g, R as above.
- Then g is analytic in $D(z_0,R)$.
- Furthermore, the derivative is given by
$$g'(z) = \sum_{j=1}^{\infty} j a_j (z-z_0)^{j-1}$$ diff term by term!

and antiderivative (up to const) is given by
$$G(z) = \sum_{j=0}^{\infty} \frac{a_j}{j+1} (z-z_0)^{j+1}$$ int. term by term!

- In particular $a_j = \frac{1}{j!} g^{(j)}(z_0)$ (diff j times term by term)

so, for a given $g(z)$ the a_j's are unique.

The key lemma is

Theorem 2 Let $\{f_n\}$, $f_n : A_{\text{open}} \to \mathbb{C}$ analytic

Suppose $\{f_n\} \to f$, uniformly on any $\overline{D(z_0,\rho)} \subset A$

Then f is analytic

and

$\{f_n'\} \to f'$, uniformly on any $\overline{D(z_0,\rho)} \subset A$

proof Let $z_0 \in A$, $\overline{D(z_0,\rho)} \subset D(z_0,\rho_1) \subset \overline{D(z_0,\rho_1)} \subset A$

then for $|z-z_0| \le \rho$,

$$f_n(z) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta \quad ; \qquad f_n'(z) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} \frac{f_n(\zeta)}{(\zeta - z)^2}\,d\zeta$$

$n \to \infty$

because $f_n \to f$ unif on γ

$f_n \to f$ unif on γ $|z-z_0| \le \rho < \rho_1$

$$f(z) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad\qquad \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$$

f is uniform limit of cont fcns on γ, so is cont; and this integral can be differentiated wrt z (we did this before!), for $|z| \le \rho$:

So $f'(z) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$ ∎

Now prove Thm 1 bullets:

- g is analytic because $S_n(z) = \sum_{j=0}^{n} a_j (z-z_0)^j \to g(z)$ uniformly on all $\overline{D(z_0,\rho)} \subset D(z_0,R)$
- $S_n'(z) = \sum_{j=0}^{n} a_j (z-z_0)^{j-1} j \to g'(z)$ uniformly on $\overline{D(z_0,\rho)} \subset D(z_0,R)$
- Define $G(z) = \sum_{j=0}^{\infty} \frac{a_j}{j+1}(z-z_0)^{j+1}$. By comparison test G has at least as large a radius of convergence (in fact the same).

  So $G'(z) = \lim \sum_{j=0}^{N} a_j (z-z_0)^j = f(z)$ ∎

Finding radius of convergence (translate $z = w - w_0$, so $z_0 = 0$)

In general, for $g(z) = \sum a_n z^n$

$$\boxed{\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

proof: if $|z| > R$, $|a_n z^n|^{1/n} = \sqrt[n]{|a_n|}\,|z|$

$\forall \varepsilon$, $\exists n$ arbitrarily large with

$$\sqrt[n]{|a_n|}\,|z| \geq \frac{1}{R+\varepsilon}|z| > 1 \quad \text{if } |z| > R+\varepsilon.$$

$\Rightarrow$ divergence.

If $|z| < R$, $\forall \varepsilon > 0\ \exists N$ s.t.

$$n \geq N \Rightarrow |a_n z^n| \leq \left(\frac{1}{R-\varepsilon}\right)^n |z|^n \to 0 \text{ geometrically if } |z| < R-\varepsilon$$

$\Rightarrow$ convergence.

So, if $\boxed{\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \text{then } R = \frac{1}{L}}$

Often, ratio test easier;

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $R = \frac{1}{L}$

proof use ratio test for series $\sum a_n z^n$

if $\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| \to \mu < 1$ conv

$\mu > 1$ div

$\downarrow$

$L|z| < 1$ conv

> 1 div

example: Find the radius of convergence for $f(z) = \sum_{n=0}^{\infty} n^2 z^n$ And cleverly find a closed form expression for f.

ans $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{n^2} \to 1$, so $R = \frac{1}{1} = 1$.

$$\frac{f(z)}{z} = \sum_{n=0}^{\infty} n\,n z^{n-1};$$

$$\int \frac{f(z)}{z} = \sum_{n=0}^{\infty} n z^n;$$

$$\frac{1}{z}\int \frac{f(z)}{z}\,dz = \sum_{n=0}^{\infty} n z^{n-1} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n = \frac{d}{dz}\left(\frac{1}{1-z}\right)$$

So $\int \frac{f(z)}{z}\,dz = \frac{z}{(1-z)^2}$

So $\frac{f(z)}{z} = \frac{(1-z)^2 + z\,2(1-z)}{(1-z)^4} = \frac{1-z^2}{(1-z)^4}$

$f(z) = \frac{z(1-z^2)}{(1-z)^4}$!? [redacted]

More neat tricks!

If $f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j$; $g(z) = \sum_{j=0}^{\infty} b_j (z-z_0)^j$ in $D(z_0, r)$

then $f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) (z-z_0)^n$ (multiply term by term)

pf $f(z)g(z)$ is analytic. at $z_0 = a_0 b_0$

$(fg)' = f'g + fg'$ at $z_0 = a_1 b_0 + a_0 b_1$

$(fg)'' = f''g + 2f'g' + fg'$ @ $z_0 = 2a_2 b_0 + 2a_1 b_1 + 2a_0 b_2$

$(fg)^{(n)}(z_0) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(z_0)\, g^{(n-j)}(z_0)$ inductive formula

$\frac{(fg)^{(n)}(z_0)}{n!} = \sum_{j=0}^{n} \cancel{\frac{1}{n!}} \left[\cancel{\binom{n}{j}} \cancel{j!}\, a_j\, \cancel{(n-j)!}\, b_{n-j} \right]$!

example $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$

$\Rightarrow \left(\frac{1}{1-z}\right)^2 = (1 + z + z^2 + \cdots)(1 + z + z^2 + \cdots)$

$= 1 + 2z + 3z^2 + 4z^3 + \cdots$

$= \sum_{j=1}^{\infty} j z^{j-1}$

$= \frac{d}{dz}\left(\frac{1}{1-z}\right)$ ✓