Math 4200
Friday 14 Oct

We will postpone Poisson Integral formula until Chp 4 (from §2.5)

Homework for Fri 10/21
3.1 4, 6, 7, 14
3.2 2, 3, 4, 7, 11, 13, 14, 18, 20

§3.1-3.2 (-33)

power series for analytic fun... more consequences of Cauchy integral formula

\[ f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{z - z_0} \, dz \]

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)(z - z_0^2)} \, dz \]

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \sum_{j=0}^{\infty} \frac{f(z)}{(z - z_0)^{j+1}} (z - z_0)^j \, dz \]

\[ f(z) = \sum_{j=0}^{\infty} \frac{f(z)}{(z - z_0)^{j+1}} (z - z_0)^j \]

 Called Taylor series, or power series.

\[ D(z_0, r) \subset A \text{ open} \]
\[ f : A \to \mathbb{C} \text{ analytic} \]
\[ \gamma = \{ |z - z_0| = r \} \]

1.

\[ \frac{1}{1 - w} = \sum_{j=0}^{\infty} w^j, \quad |w| < 1 \]

Recall (or extend from IR notions)

(a) \[ a_n = A \text{ means } \lim_{n \to \infty} S_n = A \]
for \[ S_n = \sum_{j=0}^{n} a_j \text{, the partial sums.} \]

(b) Series is convergent iff Cauchy:
\[ \forall \varepsilon > 0 \exists N \text{ s.t. } n, m > N \implies |S_n - S_m| < \varepsilon \]

(c) Series is absolutely convergent
\[ \text{iff } \sum_{j=0}^{\infty} |a_j| < \infty \]

(d) Absolute convergence \( \implies \) convergence
\[ \sum_{j=0}^{n} |a_j| \leq \sum_{j=m+1}^{\infty} |a_j| \to 0 \text{ as } n, m \to \infty \]

provided \[ \sum_{j=0}^{\infty} |a_j| < \infty \]

1 cont'd.

for geometric series
\[ S_n = \sum_{j=0}^{n} w^j \]
\[ S_n + w^{n+1} = S_{n+1} = w S_n + 1 \]
\[ S_n (1 - w) = 1 - w^{n+1} \]
\[ \frac{S_n}{1 - w} = 1 - w^{n+1} \text{ if } |w| < 1 \text{ then } S_n \to \frac{1}{1 - w} \text{ (absolutely)} \]
Interchanging \( \sum \) and \( \int \) is \( \equiv \) to

\[
\lim_{n \to \infty} \int S_n(3) \, d\gamma = \lim_{n \to \infty} \int S_n(3) \, d\gamma
\]

here \( S_n(3) = \sum_{j=0}^{n} \frac{f^{(j)}(x_j)}{(3-\gamma_j)^{j+1}} (2-\gamma_j) \)

Then if \( \{ S_n(3) \} \to S(3) \) uniformly \( \) (i.e. \( \sup_{\gamma} |S(3) - S_n(3)| \to 0 \) as \( n \to \infty \))

then \( * \) holds

\[
\left| \int S(3) \, d\gamma - \int S_n(3) \, d\gamma \right| \leq \int |S(3) - S_n(3)| \, d\gamma
\]

\[
\leq \left( \sup_{\gamma} |S(3) - S_n(3)| \right) (\text{length} \, (X))
\]

\( \to 0 \) as \( n \to \infty \), if \( S_n \to S \) uniformly.

In our case

\[
S(3) - S_n(3) = \sum_{j=n+1}^{\infty} \frac{f^{(j)}(x_j)}{(3-\gamma_j)^{j+1}} (2-\gamma_j)
\]

\[
1 \leq \sum_{j=n+1}^{\infty} \frac{|f^{(j)}(x_j)|}{r} \frac{|x_j|}{r}^j
\]

\[
\leq \left( \max_{\gamma} |f^{(j)}(\gamma)| \right) \frac{1}{r} \frac{1}{1-e^{r}}
\]

\( \downarrow \) \( 0 \), as \( n \to \infty \). Estimate is uniform!

Examples

\( f(x) = e^x \), \( x_0 = 0 \).

\( f^{(j)}(x) \big|_{x=x_0} = 1 \)

\( \Rightarrow e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \). Could also expand about other \( x_0 \)!

\( f(x) = \cos x \), \( x_0 = 0 \).

\( f^{(2j)}(x) \big|_{x=x_0} = 0 \)

\( f^{(2j+1)}(x) \big|_{x=x_0} = \begin{cases} 
0 & \text{odd} \\
(-1)^{j+1} & \text{even}
\end{cases} \)

\( \cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!} \), \( \forall x \)

\( \sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!} \)

\( f(x) = \frac{1}{1-x}, x_0 = 0 \).

\( f(x) = \sum_{j=0}^{\infty} 2^j x^j, |x| < 1 \). What about \( 121? \)
Convergence / Divergence for complex power series:

Consider \( g(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j \)

(a) If \( \exists \, z_1 \) s.t. \( \sum_{j=0}^{\infty} a_j (z-z_0)^j \) converges,

then \( \sum_{j=0}^{\infty} a_j (z-z_0)^j \) converges absolutely \( \forall \, z \) s.t. \( |z-z_0| < r \)

(and the sum is an analytic fun in \( D(z_0, r) \))

(b) If \( \exists \, z \) s.t. \( \sum_{j=0}^{\infty} a_j (z-z_0)^j \) diverges

then \( \sum_{j=0}^{\infty} a_j (z-z_0)^j \) diverges \( \forall \, z \) s.t. \( |z-z_0| > r \).

\[
\begin{align*}
(b) & \text{ follows from (a),} \\
& \text{since converge at such} \\
& \text{a } z \text{ would imply converge at } z_1. \\
\end{align*}
\]

So, if we define

\[
R := \sup \left\{ |z-z_0| \mid \text{sum converges at } z \right\}
\]

and call it the radius of convergence then

\( 12-20 \prec R \Rightarrow \text{series converges (absolutely), to an analytic function.} \)

\( 12-20 \succ R \Rightarrow \text{series diverges.} \)

proof of (a): \( \sum_{j=0}^{\infty} a_j (z-z_0)^j \) converges \( \Rightarrow \lim_{h \to \infty} a_n (z-z_0)^h = 0 \) \[ \text{Never assume the converse is true!!!} \]

\[
\Rightarrow \sup_{new} |a_n| |z-z_0|^n := M < \infty.
\]

Let \( 12-20 < r = 12-20 \)

then \( \sum_{j=0}^{\infty} |a_j (z-z_0)^j| = \sum_{j=0}^{\infty} |a_j r_j (z)| \leq \sum_{j=0}^{\infty} M (r)^j = \frac{M}{1-r} < \infty \)

so series converges absolutely.

The analyticity will follow from the fact that the convergence is uniform on any \( D(20, r) \) with \( \rho \prec r \)

(again using Cauchy!)

(Monday)