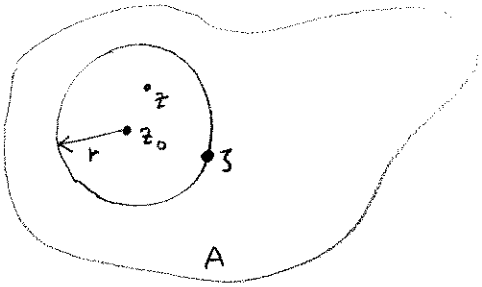
HW for Fri 10/21
3.1 4, 6, 7, 14
3.2 2, 3, 4, 7, 11, 13, 14, 18, 20

Math 4200
Friday 14 Oct

We will postpone Poisson Integral formula until Chptr 5.
(from §2.5)

§3.1-3.2 (-3.3)
power series for analytic fns... more consequences of Cauchy integral formula



$\overline{D(z_0,r)} \subset A_{open}$
$f: A \to \mathbb{C}$ analytic
$\gamma = \{|\zeta - z_0| = r\}$

$z \in D(z_0, r)$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)}\left(\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}}\right) d\zeta$$

1 geometric series

$$\sum_{j=0} \left(\frac{z-z_0}{\zeta - z_0}\right)^j$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^\infty \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} (z-z_0)^j\, d\zeta$$

2 interchange int & Σ

$$f(z) = \frac{1}{2\pi i}\sum_{j=0}^\infty \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} (z-z_0)^j\, d\zeta$$

$$= \frac{1}{2\pi i}\sum_{j=0}^\infty (z-z_0)^j \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta$$

$$\boxed{f(z) = \sum_{j=0}^\infty \frac{1}{j!} f^{(j)}(z_0)\,(z-z_0)^j \qquad |z - z_0| < r}$$

Called Taylor series, or power series.

1

$$\frac{1}{1-w} = \sum_{j=0}^\infty w^j \qquad |w| < 1$$

recall (or extend from $\mathbb{R}$ notions)

(a) $\sum_{j=0}^\infty a_j = A$ means $\lim_{n\to\infty} S_n = A$ for $S_n = \sum_{j=0}^n a_j$, the partial sums.

(b) series in convergent iff Cauchy:
$\forall \varepsilon > 0\ \exists N$ s.t. $n, m \geq N \Rightarrow |S_n - S_m| < \varepsilon$

(c) series := absolutely convergent iff $\sum_{j=0}^\infty |a_j| < \infty$

(d) absolute convergence $\Rightarrow$ convergence

pf $|S_n - S_m| = \left|\sum_{j=m+1}^n a_j\right|$ ($n > m$)
$\leq \sum_{j=m+1}^n |a_j| \to 0$ as $m, n \to \infty$
provided $\sum |a_j| < \infty$

1 cont'd:
for geometric series

$$S_n = \sum_{j=0}^n w^j$$

$$S_n + w^{n+1} = S_{n+1} = w S_n + 1$$

$$S_n(1-w) = 1 - w^{n+1}$$

$$\boxed{S_n = \frac{1-w^{n+1}}{1-w}}$$

if $|w| < 1$ then $S_n \to \frac{1}{1-w}$ (absolutely)

Interchanging Σ & int is $\equiv$ to

$$* \quad \int_\gamma \left(\lim_{n\to\infty} S_n(\zeta)\right) d\zeta = \lim_{n\to\infty} \int_\gamma S_n(\zeta)\, d\zeta \qquad \text{here } S_n(\zeta) = \sum_{j=0}^{n} \frac{f(\zeta)}{(\zeta-z_0)^{j+1}}(z-z_0)^j$$

Thm If $\{S_n(\zeta)\} \to S(\zeta)$ uniformly (i.e. $\sup_{\zeta\in\gamma} |S(\zeta)-S_n(\zeta)| \to 0$ as $n\to\infty$)

then * holds

pf
$$\left| \int_\gamma S(\zeta)\,d\zeta - \int_\gamma S_n(\zeta)\,d\zeta \right| \le \int_\gamma |S(\zeta)-S_n(\zeta)|\,|d\zeta|$$
$$\le \left(\sup_{\zeta\in\gamma} |S(\zeta)-S_n(\zeta)|\right)(\text{length}(\gamma))$$
$\to 0$ as $n\to\infty$, if $S_n \to S$ uniformly.

$\blacksquare$

In our case
$$S(\zeta)-S_n(\zeta) = \sum_{j=n+1}^{\infty} \frac{f(\zeta)}{(\zeta-z_0)^{j+1}}(z-z_0)^j$$
$$|\quad| \le \sum_{j=n+1}^{\infty} \frac{|f(\zeta)|}{r}\left|\frac{\rho}{r}\right|^j \qquad |z-z_0| = \rho < r = |\zeta - z_0|$$
$$\le \left(\max_{\zeta\in\gamma} |f(\zeta)|\right)\frac{1}{r}\left|\frac{\rho}{r}\right|^{n+1} \frac{1}{1-\rho/r}$$
$\downarrow$

0, as $n\to\infty$. Estimate is uniform!

Examples

$f(z) = e^z$, $z_0 = 0$. $\quad f^{(j)}(z)\Big|_{z=0} = 1$

$$\Rightarrow e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$$
. Could also expand about other z_0!

$f(z) = \cos z$, $z_0 = 0$ $\quad f^{(j)}(z)\Big|_{z=0} = \begin{cases} 0 & j \text{ odd} \\ (-1)^{j/2} & j \text{ even} \end{cases}$

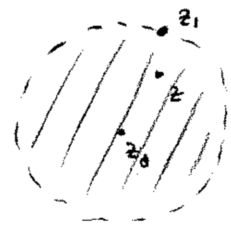
$$\cos z = \sum_{j=0}^{\infty} \frac{1}{(2j)!}(-1)^j z^{2j} \qquad \forall z$$
$$\sin z = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!}(-1)^{j+1} z^{2j+1}$$

$f(z) = \frac{1}{1-z}$, $z_0 = 0$. $\quad f(z) = \sum_{j=0}^{\infty} z^j \quad |z| < 1$. What about $|z| > 1$?

Convergence / Divergence for complex power series:

Consider

$$g(z) := \sum_{j=0}^{\infty} a_j (z-z_0)^j$$



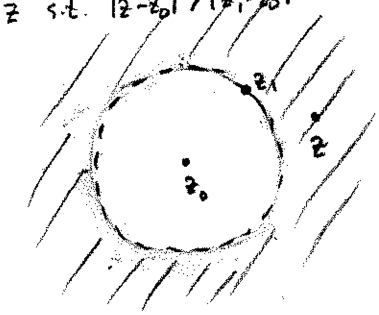
(a) If $\exists\, z_1$ s.t. $\sum_{j=0}^{\infty} a_j (z_1-z_0)^j$ converges,

then $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ converges absolutely $\forall z$ s.t. $|z-z_0| < |z_1-z_0| =: r$

(and the sum is an analytic fcn in $D(z_0, \varrho)$)

(b) If $\exists\, z_1$ s.t. $\sum_{j=0}^{\infty} a_j (z_1-z_0)^j$ diverges

then $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ diverges $\forall z$ s.t. $|z-z_0| > |z_1-z_0|$



[(b) follows from (a), since convergence at such a z would imply convergence at z_1 !]

So, if we define

$$R := \sup \left\{ |z-z_0| \text{ s.t. } \sum a_j (z-z_0)^j \text{ converges at } z \right\}$$

and call it the radius of convergence then

$|z-z_0| < R \Rightarrow$ series converges (absolutely), to an analytic function.

$|z-z_0| > R \Rightarrow$ series diverges.

proof of (a): $\sum_{j=0}^{\infty} a_j (z_1-z_0)^j$ converges $\Rightarrow \lim_{n\to\infty} a_n (z_1-z_0)^n = 0$ [Never assume the converse is true !!!]

$\Rightarrow \sup_{n\in\mathbb{N}} |a_n| |z_1-z_0|^n =: M < \infty.$

write $r := |z_1-z_0|$

let $|z-z_0| = \varrho < r = |z_1-z_0|$

then $$\sum_{j=0}^{\infty} |a_j (z-z_0)^j| = \sum_{j=0}^{\infty} |a_j| r^j \left(\frac{\varrho}{r}\right)^j \le \sum_{j=0}^{\infty} M \left(\frac{\varrho}{r}\right)^j = \frac{M}{1-\varrho/r} < \infty$$

so series converges absolutely.

The analyticity will follow from the fact that the convergence is uniform on any $\overline{D(z_0,\varrho)}$ with $\varrho < r$

(again using Cauchy!)

(Monday)