Remarks:

1. In the local max principle which preceded it, we could’ve carried out the step that \( |f(z)| \equiv M \Rightarrow f(z) \text{ constant} \) as we started doing when I didn’t follow my notes:

   \[
   |f|^2 = M^2 = u^2 + v^2
   \]

   \[
   \Rightarrow u u_x + v v_x = 0
   \]

   \[
   u u_y + v v_y = 0
   \]

   \[
   \begin{bmatrix}
   u_x & v_x \\
   u_y & v_y 
   \end{bmatrix}
   \begin{bmatrix}
   u \\
   v 
   \end{bmatrix}
   = \begin{bmatrix}
   0 \\
   0 
   \end{bmatrix}.
   \]

   Case 1: \( M = 0 \) \( \Rightarrow f = 0 \); done

   Case 2: \( M > 0 \) \( \Rightarrow \begin{bmatrix}
   u \\
   v 
   \end{bmatrix} \neq \begin{bmatrix}
   0 \\
   0 
   \end{bmatrix} \forall z \in D(2, r) \)

   \[
   \Rightarrow \det 0 = u_x v_y - u_y v_x
   \]

   \[
   = u_x^2 + u_y^2
   \]

   \[
   = v_x^2 + v_y^2
   \]

   \[
   \Rightarrow \nabla u \equiv 0 \nabla v \equiv 0
   \]

   \[
   \Rightarrow f \text{ const}
   \]

2. We worked the example on bottom of page 2 correctly; notes however were wrong: \( e^{-2z} = e^{-(e^{i\theta})} \) for \( z = e^{i\theta} \in \text{unit circle} \)

   \[
   = e^{-\left( \cos 2\theta + i \sin 2\theta \right)}
   \]

   \[
   1%_o = e^{-\cos 2\theta} \text{ max when } \cos 2\theta = -1
   \]

   i.e. \( \theta = \pm \pi/2 \) agrees with Monday's computation.

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Monday: Precise proof for existence of harmonic conjugates on simply connected domains.

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Monday: Mean value property, and consequent local max or min then, and global max & min principle for harmonic funs.
Two consequences of maximum modulus principle (for analytic fcn's)

- Modified proof of Fundamental Theorem of Algebra. (for fun!)

  Follow outline of other proofs, reduce to case of showing
  \[ p_n(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \]
  has at least one root (as before).
  Then substitute:
  \[
  f(z) = \frac{1}{p_n(z)} \text{ is entire.}
  \]

  \[
  \lim_{|z| \to \infty} |f(z)| = 0 \quad \text{(as before!)}
  \]

  So \( |f(z)| \) has a global maximum value \( M \) at \( z_0 \in \mathbb{C} \)

  \[ \Rightarrow \text{ (Global Max principle) } f(z) \text{ is constant on any bounded open connected domain containing } z_0 \Rightarrow f \text{ is const} \]

  \[ \Rightarrow p_n(z) \text{ is const} \]

  \[ \text{Schwarz Lemma} \quad \text{(we'll use later!)} \]

  \[ \text{Let } A = D(0,1) \]

  \[ f \text{ analytic on } A \]

  \[ f(0) = 0 \]

  \[ |f(z)| < 1 \text{ on } A \]

  \[ 1) \text{ if } |f(0)| < 1 \]

  \[ 2) \text{ if } |f(z)| < 1 \quad \forall z \in A \]

  \[ 3) \text{ if } |f(z)| = 1 \text{ for some } z \neq 0 \]

  \[ \text{or if } |f(0)| = 1 \]

  \[ \Rightarrow f(z) = e^{i\theta}z = c \quad , \quad |c| = 1 \]

  \[ \forall z \in A \]

  \[ \Rightarrow \text{ lemma holds} \]

  \[ (\text{Morera}) \]

  \[ \Rightarrow g \text{ analytic} \]

  \[ \text{Apply maximum modulus principle to } g : \]

  \[ \max_{|z| = r} |g(z)| = \max_{|z| = r} |f(z)| < \frac{1}{r} \]

  \[ \text{(let } r > 1 \text{)} \]

  \[ \Rightarrow \max_{|z| = 1} |g(z)| < 1 \Rightarrow (1),(2). \]

  \[ \text{If } \exists z \in A \text{ s.t. } |g(z)| = 1 \]

  \[ \text{deduce } g \text{ is const} \Rightarrow (3) \]