Math 4200
Wed 12 Oct

- Prove global maximum modulus principle, pages 2-3 Monday notes

Remarks:

(1) In the local max principle which preceded it, we could've carried out the step that $|f(z)| \equiv M \Rightarrow f(z)$ constant as we started doing when I didn't follow my notes:

$$|f|^2 = M^2 = u^2+v^2$$
$$\Rightarrow uu_x + vv_x = 0$$
$$uu_y + vv_y = 0$$

$$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Case 1: $M = 0 \Rightarrow f = 0$; done ∎

Case 2: $M > 0 \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \forall z \in D(z_0, r)$

$$\Rightarrow \det \equiv 0 = u_x v_y - u_y v_x$$
$$= u_x^2 + u_y^2$$
$$= v_y^2 + v_x^2$$
$\Rightarrow \nabla u \equiv 0$, $\nabla v \equiv 0$
$\Rightarrow f$ const ∎

(2) We worked the example on bottom of page 2 correctly; notes however were wrong.

$$e^{-z^2} = e^{-(e^{2i\theta})} \quad \text{for } z = e^{i\theta} \in \text{unit circle}$$
$$= e^{-[\cos 2\theta + i\sin 2\theta]};$$

$$|\%| = e^{-\cos 2\theta} \quad \text{max when } \cos 2\theta = -1$$
i.e. $\theta = \pm \pi/2$ agrees with Monday' computation

- page 4 Monday: Precise proof for existence of harmonic conjugates on simply connected domains

- Mean value property, and consequent local max or min thm, and global max & min principle for harmonic fcns, page 5 Monday

Two consequences of maximum modulus principle (for analytic fcns)

- Modified proof of Fundamental Theorem of Algebra. (for fun!)

Follow outline of other proof, reduce to case of showing

$$p_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

has at least one root (as before).

Then substitute:

If not, let $f(z) = \frac{1}{p_n(z)}$ is entire.

$$\lim_{|z|\to\infty} |f(z)| = 0 \qquad \text{(as before!)}$$

So $|f(z)|$ has a global maximum value M at $z_0 \in \mathbb{C}$

$\Rightarrow$ (Global Max principle) $f(z)$ is constant on any bounded open connected domain containing z_0 $\Rightarrow$ f is const

$\Rightarrow p_n(z)$ is const

but $p_n^{(n)}(z) = n! \neq 0$ (n^{th} deriv) $\Rightarrow\!\!\Leftarrow$.

(so p_n is not const.)

- <u>Schwarz lemma</u>: (we'll use later!)

$$\left.\begin{array}{l} \text{Let } A = D(0,1) \\ f \text{ analytic on } A \\ f(0) = 0 \\ |f(z)| \le 1 \text{ on } A \end{array}\right\} \text{ Then } \begin{array}{l} 1)\ |f'(0)| \le 1 \\ 2)\ |f(z)| \le |z| \quad \forall z \in A \\ 3)\ \text{If } |f(z)| = |z| \text{ for some } z \neq 0 \\ \quad\ \text{or if } |f'(0)| = 1 \\ \quad\ \text{Then } f(z) = e^{i\theta}z = cz,\ |c| = 1 \\ \quad\ \forall z \in A. \end{array}$$

<u>proof</u>: Let $g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$

g is analytic on $D \setminus \{0\}$ and continuous on D $\Rightarrow$ rectangle lemma holds $\Rightarrow$ g analytic

((Morera;

Apply maximum modulus principle to g:

$$\max_{|D(0,r)|} |g(z)| = \max_{|z|=r} |g(z)| \le \frac{1}{r}$$

Let $r \nearrow 1$; $\Rightarrow \max_{A} |g(z)| \le 1 \Rightarrow$ (1), (2).

If $\exists\, z_0 \in A$ s.t. $|g(z_0)| = 1$

deduce g is const $\Rightarrow$ (3)