Math 4200

Mon 10/10 (Maybe partly Wed too!)

- Morera's Thm p 5 Wed.

Some more consequences of Cauchy Integral formula (§2.5)

Mean Value Property Let f analytic in $A_{\text{open, connected}}$, $\overline{D(z_0,r)} \subset A$ (closure)

Then

(1) $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$ ⟵ f at the center of a disk is its average value over the bounding circle

(2) $f(z_0) = \frac{1}{\pi r^2}\iint\limits_{D(z_0,r)} f(x+iy)\,dx\,dy$ ⟵ and over the entire disk

pf C.I.F. $\Rightarrow f(z_0) = \frac{1}{2\pi i}\oint\limits_{|z|=r} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta$ $\qquad \zeta = z_0 + re^{i\theta}$, $d\zeta = rie^{i\theta}d\theta$; $\zeta - z_0 = re^{i\theta}$

(1) $$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, r i e^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$$

(2): $$\iint\limits_{D(z_0,r)} f(x+iy)\,dx\,dy = \int_0^r\int_0^{2\pi} f(z_0+\rho e^{i\theta})\underbrace{\rho\,d\theta\,d\rho}_{\text{"}dA\text{" in polar coords}}$$

$$= \int_0^r \rho\,(\underbrace{2\pi f(z_0)}_{})\,d\rho$$

$\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta = 2\pi f(z_0)$ by (1)!

$$= 2\pi f(z_0)\,\frac{\rho^2}{2}\Big]_0^r = f(z_0)\,\pi r^2 \qquad \blacksquare$$

Corollary: (Local Maximum modulus principle)

Let f, A as above, $\overline{D(z_0,r)} \subset A$

Suppose $|f(z_0)| := M = \max\limits_{z\in \overline{D(z_0,r)}} |f(z)|$ $\qquad$ ($|f(z)|$ has a local max at z_0).

Then in fact $f(z) \equiv f(z_0)$, is constant in $\overline{D(z_0,r)}$

proof By (2),

$$* \quad M = |f(z_0)| \le \frac{1}{\pi r^2}\iint\limits_{D(z_0,r)} |f(x+iy)|\,dA \le \frac{1}{\pi r^2}\iint\limits_D M\,dA = M$$

so all inequalities in * are actually equalities.

since f is continuous on $\overline{D(z_0,r)}$ deduce $|f(z)| = M$ $\forall z \in \overline{D(z_0,r)}$. Why?

Answer to "Why?"

if $\exists\ z_1 \in \overline{D(z_0,r)}$ with $|f(z_1)| < M-\varepsilon$

then $\exists\ z_2 \in D(z_0,r)$

$|f(z_2)| < M - \varepsilon/2$

$\Rightarrow \exists\ \varrho$ s.t. $D(z_2,\varrho) \subset D(z_0,r)$

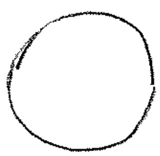
AND $|f(z)| < M - \varepsilon/4 \quad \forall z \in D(z_2,\varrho)$

$$\Rightarrow \iint_{D(z_0,r)} |f(z)|\,dA = \iint_{D(z_2,\varrho)} + \iint_{D(z_0,r)\setminus D(z_2,\varrho)}$$

$$< (M - \varepsilon/4)\,\pi\varrho^2 + M(\pi r^2 - \pi\varrho^2)$$

$$= M\pi r^2 - \frac{\varepsilon}{4}\pi\varrho^2 < M\pi r^2 \ !$$

So, $|f(z)| \equiv M$ on $\overline{D(z_0,r)}$. If $M=0$ deduce $f \equiv 0$.



Else, $f = u + iv$

$|f|^2 = M^2 = u^2 + v^2 \Rightarrow \begin{array}{l} uu_x + vv_x \equiv 0 \\ uu_y + vv_y \equiv 0 \end{array}$

$$\text{CR} \rightarrow \begin{bmatrix} u & v \\ v & -u \end{bmatrix}\begin{bmatrix} u_x \\ v_x \end{bmatrix} \equiv 0 \quad \Longrightarrow \begin{array}{l} u_x \equiv 0 \\ v_x \equiv 0 \end{array} \overset{\text{CR}}{\Rightarrow} \begin{array}{l} u_y \equiv 0 \\ v_y \equiv 0 \end{array}$$

$\uparrow$

$\det = -u^2 - v^2 = -M^2 \neq 0$

$\Rightarrow f' \equiv 0$

$\Rightarrow f \equiv f(z_0)$ ∎

Global Maximum principle

Let A be open, connected, and bounded.
Let f be analytic in A and continuous on the closure of A, $\bar{A}$.
Then $|f(z)|$ attains its maximum value on the boundary of A, ∂A.
If $\exists\ z_0 \in A$ at which $|f(z)|$ attains its max value, then $f \equiv f(z_0)$ is constant

example Find maximum of e^{-z^2} on $\overline{D(0,1)}$:

need only check on $|z|=1$, the bdry.

$$e^{-e^{2i\theta}} = e^{-(\cos 2\theta + i\sin 2\theta)}$$

$|\ | = e^{-\cos 2\theta}$ max at $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$.

value is 1

proof of global max principle

From analysis, $|f(z)|$ attains its max on $\bar{A}$. We need only check that if $|f(z)|$ has an interior max, then $f(z)$ is constant!

So, let $z_0 \in A$, $|f(z_0)| = M = \max_{\bar{A}} |f(z)|$

proof 1 : Let $B = \{ z \in A \mid f(z) = f(z_0) \}$

B is closed because $B = f^{-1}(\{f(z_0)\})$ is the inverse image of a closed set

(or, let $\{z_k\} \subset B$, $\{z_k\} \to z$ then $f(z) = \lim f(z_k) = M$ so $z \in B$)

B is open by the local maximum modulus thm

B is nonempty by construction

A connected $\Rightarrow B = A$ (by one def. of connected)

proof 2 Let $z \in A$. Let $\gamma : [0,1] \to A$ continuous path (by def. of path connected)

$\gamma(0) = z_0$

$\gamma(1) = z$

Let $t = \sup \{ s \mid f(\gamma(r)) = f(z_0) \ \forall\, 0 \le r \le s \}$. Note $t > 0$

If $t = 1$ then $f(\gamma(s)) = f(z_0)$ $\forall\, 0 \le s < t$

$\lim_{s \to t} \Rightarrow f(z) = f(z_0)$

t must $= 1$, since $f(\gamma(s)) = f(z_0)$ $\forall\, s < t$

$\lim_{s \to t} \Rightarrow f(\gamma(t)) = f(z_0)$

Then local max princ $\Rightarrow f(D(\gamma(t), \epsilon)) \equiv f(z_0)$

which would contradict $t < 1$. ■

What we can do for analytic we can do for harmonic!

Lemma: If A is open, simply connected, and if $u \in C^2$ is harmonic then $u = \mathrm{Re} f$ for some analytic function $f = u + iv$ defined on A

(This replaces our earlier hazy claim that harmonic conjugates exist locally. We could actually prove this lemma by repeating steps from §2.2, i.e. rectangle lemma, harmonic conjugates in disks, deformation thm ....) It is more elegant to use all of that hard work w/o repeating it:

Let $g := u_x - iu_y$
$\quad := U + iV$

(Note, if $f = u + iv \; \exists$, $f' = f_x = u_x + iv_x = u_x - iu_y$).

g is analytic:

$$\left.\begin{array}{l} \text{CR: } U_x = u_{xx} \overset{u \text{ harmonic}}{=} -u_{yy} = V_y \\ \phantom{\text{CR: }} U_y = u_{xy} \underset{u\, C^2}{=} u_{yx} = -V_x \\ \& \; U, V \text{ are } C^1 \end{array}\right\} \Rightarrow g \text{ is analytic.}$$

A is simply connected $\Rightarrow \exists$ antiderivative, call it $f(z)$

then $f'(z) = f_x = -if_y$

$$\Rightarrow U + iV = (\mathrm{Re} f)_x + i(\mathrm{Im} f)_x = -i\left((\mathrm{Re} f)_y + i(\mathrm{Im} f)_y\right)$$

$\Rightarrow (\mathrm{Re} f)_x = U = u_x$
$\Rightarrow (\mathrm{Re} f)_y = -V = u_y$;

$$\left.\begin{array}{l}(\mathrm{Re} f)_x = u_x \\ (\mathrm{Re} f)_y = u_y\end{array}\right\} \Rightarrow \mathrm{Re} f = u + c$$

$\Rightarrow \mathrm{Im} f$ is harmonic conjugate. ∎

<u>Corollary</u> (See page 1)

Let u be C^2 and harmonic in A open, connected, $\overline{D(z_0,r)} \subset A$

Then

(1) $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$

(2) $u(z_0) = \frac{1}{\pi r^2}\iint\limits_{D(z_0,r)} u(x+iy)\,dx\,dy$

pf on $\overline{D(z_0,r)} \subset D(z_0,r_1) \subset A$ $\exists\, v$ s.t. $u+iv = f(z)$ is analytic.
Apply MVP from page 1 & take real parts.

<u>Corollary</u>: (See pages 1-3)

<u>Local max or min thm</u>: Let $u \in C^2$ harmonic on an open set containing $\overline{D(z_0,r)}$

Let $m := \min\limits_{z \in \overline{D(z_0,r)}} u(z) \le \max\limits_{z \in \overline{D(z_0,r)}} u(z) := M$

If $u(z_0) = m$ or $u(z_0) = M$ then u is constant on $\overline{D(z_0,r)}$

pf repeat analogous chain to $*$ on page 1, <u>without</u> absolute value signs

<u>Global max or min thm</u>: Let A open, connected, bounded. Let $u \in C(\overline{A})$ and $u \in C^2(A)$ and u harmonic.

then u attains its max & min values on ∂A. If either the max or min is attained at an interior point then u is constant.

pf: See pages 2-3!