Math 4200
Wed 9 Nov.

- Please replace page 1 of Monday's notes with the expanded version, which explains why we can weaken the hypotheses on the series expansion theorem from

$$\lim_{|z|\to\infty} |z||f(z)| = 0$$

to

$$\limsup_{|z|\to\infty} |z||f(z)| \le M.$$

- Decide if midterm is still Monday next week (11/14)
  - Review sheet at end of today's notes
  - 2002 Exam (& sol'ns) posted on our web page, for practice
- Do example 4 in Monday (11/7's) notes.

- If time permits, begin Laplace transform discussion from chapter 8:
  There is a formula for $L^{-1}(F(s))(t) = f(t)$ involving the residue theorem:

Remember this from 2280/2250? You always used a table to find $L^{-1}(F(s))(t)$!

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\frac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$	$\left[\!\left[\frac{t}{a}\right]\!\right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		

Laplace Transform: $f: [0,\infty) \to \mathbb{C}$

$$* \quad (\mathcal{L}f)(z) := \int_0^\infty e^{-zt} f(t)\,dt$$

$\mathcal{L}$ is linear operator:
$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$$

We restrict to f of exponential order, i.e.

$\exists\ A, B \in \mathbb{R}$ s.t.

$$|f(t)| \leq A e^{Bt} \quad \forall t.$$

Then, for $\operatorname{Re} z > B$, $*$ converges absolutely, since

$$\left| \int_0^N e^{-zt} f(t)\,dt \right| \leq \int_0^N |e^{-zt} f(t)|\,dt \leq \int_0^N e^{-t(\operatorname{Re} z)} A e^{Bt}\,dt$$
$$= A \int_0^N e^{t(B - \operatorname{Re} z)}\,dt < A \int_0^\infty \text{″} = \frac{A}{\operatorname{Re} z - B}$$

Theorem: If $f: [0,\infty) \to \mathbb{C}$ is of exponential order, then $\exists!\ \sigma \in (-\infty, \infty)$ s.t. $*$ converges for $\operatorname{Re} z > \sigma$
diverges for $\operatorname{Re} z < \sigma$

And on $A := \{ z \in \mathbb{C} \text{ s.t. } \operatorname{Re} z > \sigma \}$

$F(z) = (\mathcal{L}f)(z)$ is analytic, with

$$F'(z) = \int_0^\infty -t e^{-zt} f(t)\,dt$$

idea of proof:

This is like radius of conv. thm: If integral converges for z with $\operatorname{Re} z = \sigma_1$

then it will converge absolutely uniformly for $\operatorname{Re} w \geq \sigma_2 > \sigma_1$

If you define $F_N(z) = \int_0^N e^{-zt} f(t)\,dt$

then $F_N'(z) = \int_0^N -t e^{-zt} f(t)\,dt$

$F_N \to F$ uniformly on $\{ w \mid \operatorname{Re} w \geq \sigma_2 \}$

$\Rightarrow F$ analytic on $\{ w \mid \operatorname{Re} w > \sigma_2 \}$. ∎

example

$$f(t) = e^{at}$$
$$F(z) = \int_0^\infty e^{-zt} e^{at}\,dt = \left. \frac{e^{(a-z)t}}{a-z} \right]_0^\infty = \frac{1}{z-a} \quad \operatorname{Re} z > a.$$
diverges $\operatorname{Re} z < a$

$\sigma(f) = a$

Note also, $\mathcal{L}\{t e^{at}\} = -\frac{d}{dz} \frac{1}{(z-a)} = \frac{1}{(z-a)^2}$

$\mathcal{L}\{t^2 e^{at}\} = \frac{2}{(z-a)^3}$ etc.

Inverse Laplace transform

Suppose $F(z)$ is analytic on $\mathbb{C}$ except for a finite number of (isolated) singularities

Suppose $\exists\ \sigma \in \mathbb{R}$ s.t. F is analytic $\forall z$ s.t. $\operatorname{Re} z > \sigma$

Suppose $\lim_{z \to \infty} F(z) = 0$

Then

$$(\mathcal{L}^{-1}F)(t) := f(t) = \sum_{z_j \text{ sing of } F} \operatorname{res}\left(e^{zt}F(z),\ z_j\right)$$

examples $F(z) = \frac{1}{(z-a)^2}$:

$$f(t) = \operatorname{res}\left(e^{zt}/(z-a)^2,\ a\right)$$

$$\boxed{= t e^{at}}$$

$$e^{zt} = e^{at} e^{(z-a)t} = \frac{e^{at}\left(1 + (z-a)t + \dots\right)}{(z-a)^2}$$

or

$F(z) = \frac{1}{z^2+a^2}$:

$$f(t) = \operatorname{res}\left(\frac{e^{zt}}{z^2+a^2},\ ai\right) + \operatorname{res}\left(\frac{e^{zt}}{z^2+a^2},\ -ai\right)$$

$$\frac{e^{zt}}{(z-ai)(z+ai)}$$

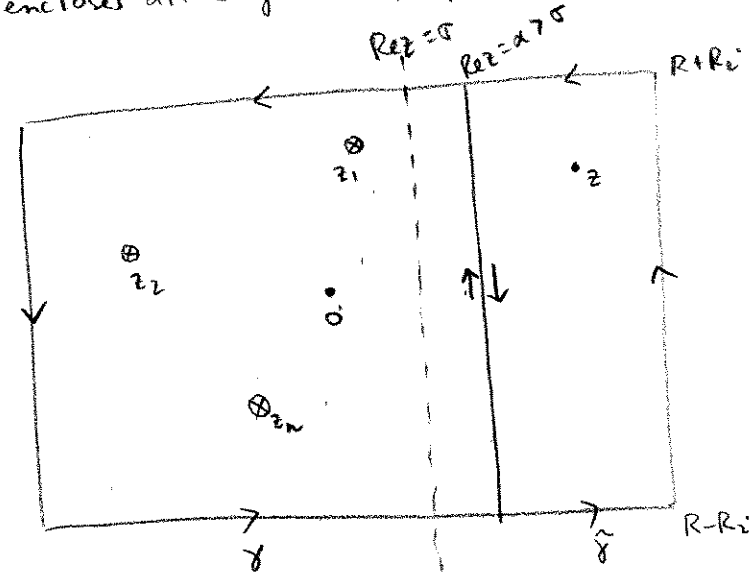
$$= \frac{e^{ait}}{2ai} + \frac{e^{-ait}}{-2ai} = \frac{1}{a}\,\frac{1}{2i}\left(e^{ait} - e^{-ait}\right)$$

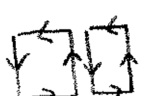
$$\boxed{= \frac{1}{a}\sin at}$$

proof of inversion formula :

We will take $f(t)$ as defined in the thm & show $\mathcal{L}f = F$. (eventually).

We can get $f(t)$ from the residue thm, by picking the curve γ below which encloses all singularities of $F(z)$





$\gamma + \tilde{\gamma} := \Gamma$

so $f(t) = \frac{1}{2\pi i}\int_\gamma e^{\zeta t}F(\zeta)\,d\zeta$ by Residue Thm.

so $\mathcal{L}\{f\}(z) = \lim_{N\to\infty} \int_0^N e^{-zt} f(t)\,dt$

$$= \lim_{N\to\infty} \frac{1}{2\pi i}\int_0^N \int_\gamma e^{-zt} e^{\zeta t} F(\zeta)\,d\zeta\,dt$$

$$= \lim_{N\to\infty} \frac{1}{2\pi i}\int_\gamma \int_0^N e^{t(\zeta - z)} F(\zeta)\,dt\,d\zeta$$

[interchange order of integration → the contour int is really just a ds integral, if $\zeta: I \to \mathbb{C}$, $\zeta(s)$.]

$$= \lim_{N\to\infty} \frac{1}{2\pi i}\int_\gamma F(\zeta) \left.\frac{e^{t(\zeta-z)}}{\zeta - z}\right]_{t=0}^{N} d\zeta$$

$$* \quad = -\frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)}{\zeta - z}\,d\zeta$$

provided $\operatorname{Re} z > \operatorname{Re}\zeta$ on γ, i.e. $\operatorname{Re} z > \alpha$.

since $\gamma + \tilde{\gamma} = \Gamma$

$$* \quad = \underbrace{\frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{F(\zeta)}{\zeta - z}\,d\zeta}_{\substack{F(z) \\ \text{C.I.F.}}} - \underbrace{\int_\Gamma \frac{F(\zeta)}{\zeta - z}\,d\zeta}_{\downarrow\ 0}$$

as $R\to\infty$: $\left|\int_\Gamma \frac{F(\zeta)}{\zeta - z}\,d\zeta\right| \le 8R \max_\Gamma \left|\frac{F(\zeta)}{\zeta - z}\right|$

$$\le 8R\left(\frac{1}{R - |z|}\right) \max_\Gamma |F(\zeta)|$$

$8R\left(\frac{1}{R-|z|}\right) \xrightarrow{R\to\infty} 8$, $\quad \max_\Gamma |F(\zeta)| \xrightarrow{R\to\infty} 0$ by hypothesis!

__Cool example__

Let $F(z) = \frac{P(z)}{Q(z)}$ P, Q polys, $\deg P < \deg Q$

suppose Q has simple zeros, ie.

$$Q(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)$$

z_j distinct.

Then $\operatorname{Res}(F, z_j) = \frac{P(z_j)}{Q'(z_j)}$

$$\Rightarrow f(t) = \sum_j e^{z_j t}\,\frac{P(z_j)}{Q'(z_j)}$$

In general, you can prove the algebraic fact that the partial fractions algorithm works using Laplace tranform and its inverse! (or use linear algebra)

In 2280 you would've done this by partial fractions:

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^n \frac{c_j}{(z - z_j)}$$

$$\Rightarrow f(t) = \sum c_j e^{z_j t}$$

So it must be that $c_j = \frac{P(z_j)}{Q'(z_j)}$

__example__ $\frac{1}{z^4 - 1} = \frac{1}{(z-i)(z+i)(z-1)(z+1)}$

$Q'(z) = 4z^3$

so
$$\frac{1}{z^4-1} = \frac{1}{4i^3}\frac{1}{z-i} + \frac{1}{4(-i)^3}\frac{1}{z+i} + \frac{1}{4}\frac{1}{z-1} + \frac{1}{4(-1)^3}\frac{1}{z+1}$$

$$= \frac{1}{4}\left[\frac{i}{z-i} - \frac{i}{z+i} + \frac{1}{z-1} - \frac{1}{z+1}\right]$$

e.g. $f(z) = \frac{1}{z^2-4}$

Math 4200
2^{nd} Exam Review Sheet; §2.4 – 4.4

Concepts diagram: You should know key definitions & thms (statements & proofs)

§2.3
Deformation Thm for contour integrals
(only need to know statement for this exam)

§2.4
Cauchy Integral Formula & Index

$f(z)I(\gamma;z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$

C.I.F. for derivatives & resulting estimates
(f analytic $\Rightarrow$ f is C^∞, with estimates)

Morera's Thm
10/5 page 5

Liouville Thm

Fundamental Thm of Algebra.

$\exists$ harmonic conjugates on simply connected domains

§2.5
Mean value property (harmonic & analytic

Maximum modulus principles

3220, §3.1
convergence of series of numbers and functions
... absolute, uniform convergence tests
(ratio, comparison...)

$\frac{1}{1-w} = \sum_{j=0}^{\infty} w^j \;, \; |w|<1$

uniform limits of analytic fcns are analytic

§3.2
Power series for analytic fcns
& vise versa
radius of conv.
Taylor's thm.
cloning thms
multiplying power series
differentiating & integrating.

§3.3
Laurent series for analytic fcns in annuli
classification of singularities
"poles" and the Riemann sphere
multiplying Laurent series

§4.2
Residue thm

§4.1
computing residues

§4.3
Evaluation of definite integrals via contour integrals

§4.4
summing series via "$f(z)\pi\cot\pi z$"

Format of exam will be like last time - probably choose 4 out of 6.

Expect
- some problems which are key theorems i.e. prove every fcn analytic near z_0 has a power series there
- some contour integrals
- computing some residues and/or Laurent/Taylor
- more!