Math 4200  

Monday, Nov. 7  

After today's examples, we will use contour integration and residue theorem to revisit Laplace transform, and then to study various classical PDE techniques.

- magic series summation, pages 3-4 Friday notes. (Example 3)

- Same idea leads to infinite series expressions for various functions (see e.g. HW)

Notice that sufficient conditions in series example, so that

\[
\int_{\gamma_n} f(z) \cot \pi z \, dz \to 0 \quad \text{as} \quad n \to \infty
\]

is that

\[
\lim_{n \to \infty} \left| \int_{\gamma_n} f(z) \cot \pi z \, dz \right| = 0
\]

(Since \(|\cot \pi z| < 2\pi|z|^{n-1}\) on \(\gamma_n, n \text{ large}\),

so \(\left| \int_{\gamma_n} f(z) \cot \pi z \, dz \right| \leq \frac{1}{\pi} \max_{1 \leq |z| \leq 2} |f(z)| \frac{2\pi}{n} \to 0\))

So, for example, if

\[ f(z) = \frac{1}{2-z}, \quad z \notin \mathbb{Z}, \quad |z| > 1 \]

\[ \text{deduce } \lim_{n \to \infty} \left( \sum_{j=-n}^{n} \frac{1}{n-j} + \pi \cot \pi z_0 \right) = 0 \]

i.e.

\[ \pi \cot \pi z = \lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{n-j} = \frac{1}{2} + \lim_{n \to \infty} \sum_{j=0}^{n} \frac{2\pi}{z^{2}-j^{2}} \]

Need weaker

**Theorem (p 808)**

If \( \exists R > 0 \) s.t. if analytic outside \( D(0,R) \), and \( |f(z)| \leq M \) there, then

Then \( \int_{\gamma_n} f(z) \cot \pi z \, dz \to 0 \) as \( n \to \infty \)

**proof:** \( |f(z)| \leq \frac{M}{2R} \) for \( 1 \leq |z| \leq R \) - Laurent series for

\[ f(z) = \frac{b_1}{2} + \frac{b_2}{z^2} + \ldots - \frac{b_3}{z^3} + \ldots \]

abs. unif conv., \( 1 \leq |z| \leq R \).

But \( \int_{\gamma_n} \frac{1}{2} \pi \cot \pi z \, dz = 0 \), so \( \int_{\gamma_n} f(z) \pi \cot \pi z \, dz = \int_{\gamma_n} f(z) \pi \cot \pi z \, dz \)

use residue theorem, or fact that \( \frac{1}{2} \pi \cot \pi z \) is even function.
**Example 4**

\[
\int_0^\infty \frac{\sin x}{x} \, dx
\]

**Stirling's Approximation**

\[
\Pi \left( \frac{2n+1}{2n} \right) \sim \frac{2n}{e} \sqrt{\frac{2n}{\pi}}
\]

...alternating decreasing term series, so convergence at \( \infty \) is no problem, i.e.

\[
\lim_{n \to \infty} \int_0^n \frac{\sin x}{x} \, dx = \int_0^\infty \frac{\sin x}{x} \, dx
\]

\[= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \text{ since } \frac{\sin x}{x} \text{ is even.}\]

As in example 3 want to use \( e^{ix} \) rather than \( \sin x \).

Notice that \( \frac{e^{ix}}{x} \) has trouble near \( x = 0 \) however (in the real part).

In this example we could just focus on \( \text{Im part} \), but also a good opportunity to introduce principal value.

If \( f \) has an isolated, (real) singularity at \( x_0 \in \mathbb{R} \), \( x_0 \in [a,b] \)

then

\[
P.V. \int_a^b f(x) \, dx := \lim_{\epsilon \to 0} \left( \int_a^{x_0-\epsilon} f(x) \, dx + \int_{x_0+\epsilon}^b f(x) \, dx \right)
\]

notice the symmetric way in which \( x_0 \) is avoided.

**Example**

\[
\int_{-\infty}^{\infty} \frac{\cos x}{x} \, dx
\]

doesn't strictly make sense, since e.g.,

\[
\int_0^\infty \frac{\cos x}{x} \, dx = \pi \ln 2
\]

but

\[
P.V. \int_{-\infty}^{\infty} \frac{\cos x}{x} \, dx
\]

\[= \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-\varepsilon} \frac{\cos x}{x} \, dx - \int_{\varepsilon}^{\infty} \frac{\cos x}{x} \, dx \right)
\]

exactly cancel!

since \( \frac{\cos x}{x} \) is odd.

\[= 0,\]

\[\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} \]
Thus \[ \int_0^\infty \frac{\sin x}{x} \, dx = \frac{1}{2} \text{Im} \left( \right) \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx \]
\[ = \frac{1}{2} \text{Im} \lim_{\varepsilon \to 0} \left( \lim_{R \to \infty} \left( \int_{-\varepsilon}^{\varepsilon} \frac{x}{e^x} \, dx + \int_{-R}^{R} \frac{e^x}{x} \, dx \right) \right) \]

\[ \int_{\gamma_1} f(z) \, dz = 0, \]
\[ \lim_{R \to \infty} \int_{\gamma_2} f(z) \, dz = 0. \]

\[ \int_{\gamma_3} \frac{e^{iz}}{z} \, dz = \int_{2}^{z} \frac{1 \, d\varepsilon}{\varepsilon} + \text{analytic} \quad \text{as} \quad z \to 0 \]
\[ = \int_{2}^{z} \frac{1 \, d\varepsilon}{\varepsilon} + M \cdot \varepsilon \quad \text{as} \quad \varepsilon \to 0 \]
\[ = -\pi i + M\varepsilon \quad \text{as} \quad \varepsilon \to 0 \]

So \[ \int_{\gamma_1} f(z) \, dz + \int_{\gamma_3} f(z) \, dz = + \int_{\gamma_2} f(z) \, dz - \int_{\gamma_4} f(z) \, dz \]
\[ \lim_{R \to \infty} \int_{\gamma_2} f(z) \, dz + \int_{\gamma_4} f(z) \, dz = \pi i + M\varepsilon \]

\[ \lim_{R \to \infty} \int f(z) \, dz = \pi i \]

\[ \text{PV} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx = \pi i \]

\[ \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{1}{2} \pi \]