Math 4200
Monday 7 Nov.

: After today's examples, we will use contour integration and residue theorem to revisit Laplace transform, and then to study various classical PDE techniques.

- magic series summation, pages 3-4 Friday notes. (Example ③)

- Same idea leads to infinite series expressions for various functions (see e.g. HW)

Notice that sufficient conditions in series example, so that

$$\int_{\gamma_n} f(z)\pi\cot\pi z\, dz \longrightarrow 0 \quad \text{as } n\to\infty$$

is that $\lim_{|z|\to\infty} |z||f(z)| = 0$

(Since $|\pi\cot\pi z| \le 2\pi$ on γ_n, n large,

so $\int_{\gamma_n} |z||f(z)||\pi\cot\pi z|\frac{|dz|}{|z|}$

$$\le \underbrace{\left(\max_{|z|\ge n} |z||f(z)|\right)}_{\downarrow\, 0}\underbrace{\left(\frac{1}{n}4(n+1)\right)}_{\downarrow\, 4} \to 0 \Bigg)$$

So, for example, if

$f(z) = \frac{1}{z-z_0}$ $\quad z_0\notin\mathbb{Z}$, $|z_0|<n$:

deduce
$$\lim_{n\to\infty}\left(\sum_{j=-n}^{n}\frac{1}{n-z_0} + \underset{\text{res}(f(z)\pi\cot\pi z,\, z_0)}{\underset{\uparrow}{\pi\cot\pi z_0}}\right) = 0$$

i.e.
$$\pi\cot\pi z = \lim_{n\to\infty}\sum_{j=-n}^{n}\frac{1}{z-j} = \frac{1}{z} + \lim_{n\to\infty}\underset{\substack{\uparrow\\ \text{converges absolutely.}}}{\sum_{j=0}^{n}\frac{2z}{z^2-j^2}}$$

Need weaker hyp:

<u>Theorem</u> (p 308)

If $\exists$ $R>0$, $M>0$ s.t. f analytic outside $D(0,R)$, and $|z||f(z)|\le M$ there,

Then $\int_{\gamma_n} f(z)\pi\cot\pi z\, dz \to 0$ as $n\to\infty$

book also writes this in a clearly convergent way as

$$\pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

(uniformly absolutely on compact subsets avoiding the integers)

<u>proof</u>: $|f(z)|\le\frac{M}{|z|}$ for $|z|>R \Rightarrow$ Laurent series for

$$f(z) = \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \frac{b_j}{z^j} + \cdots$$

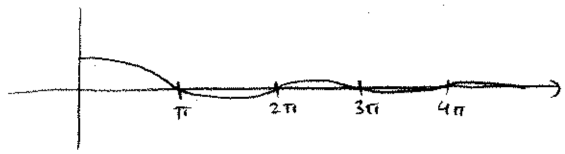
(from integral formula for coeff's)

abs. unif conv, $|z|>2R$.

But $\int_{\gamma_n}\frac{1}{z}\pi\cot\pi z\, dz \underset{\substack{\uparrow\\ \text{use residue thm,}\\ \text{or fact that } \frac{1}{z}\cot\pi z \text{ is even fcn}}}{=} 0$, so $\int_{\gamma_n} f(z)\pi\cot\pi z\, dz = \int_{\gamma_n}\underbrace{\left(f(z)-\frac{b_1}{z}\right)}_{|f(z)-\frac{b_1}{z}|\le\tilde{M}\frac{1}{|z|^2}\text{ as }|z|\to\infty}\pi\cot\pi z\, dz \to 0$ by unimproved result. ▨

• example 4

$$\int_0^\infty \frac{\sin x}{x}\,dx$$



$$= \sum_{n=0}^{\infty} \underbrace{\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx}$$

alternating decreasing term series, so convergence at ∞ is no problem, i.e.

$$\lim_{N\to\infty} \int_0^N \frac{\sin x}{x}\,dx \ \exists.$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \qquad \text{since } \frac{\sin x}{x} \text{ is even.}$$

As in example 2 want to use e^{ix} rather than $\sin x$.
Notice that $\frac{e^{ix}}{x}$ has trouble near $x=0$ however (in the real part).

in this example we could just focus on Im part, but, also a good opportunity to introduce "principal value"

If f has an isolated (real) singularity at $x_0 \in \mathbb{R}$, $x_0 \in [a,b]$
then

$$\text{P.V.} \int_a^b f(x)\,dx := \lim_{\varepsilon\to 0} \int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^{b} f(x)\,dx$$

notice the symmetric way in which x_0 is avoided.

example $\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx$ doesn't strictly make sense, since e.g.

$$\int_0^\varepsilon \frac{\cos x}{x}\,dx = +\infty$$

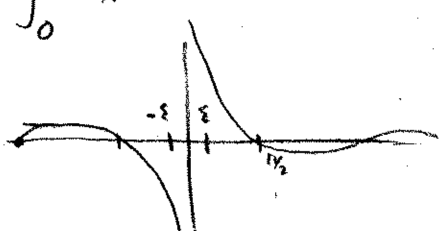
but

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx$$

$$= \lim_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} \frac{\cos x}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\cos x}{x}\,dx$$

exactly cancel! since $\frac{\cos x}{x}$ is odd

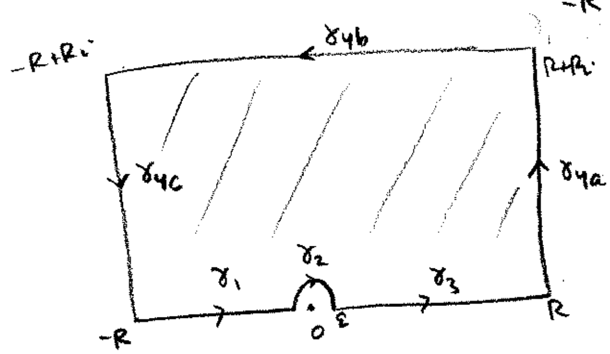
$$= 0.$$



Thus $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\,\mathrm{Im.}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$

$$= \frac{1}{2}\,\mathrm{Im}\ \lim_{\varepsilon\to 0}\left(\lim_{R\to\infty}\left(\int_{-R}^{\varepsilon}\frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R}\frac{e^{ix}}{x}\,dx\right)\right)$$

$$\searrow \int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$



$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\,dz = 0.$$

$$\lim_{R\to\infty}\int_{\gamma_4} f(z)\,dz = 0.$$

γ_{4a}: $\left|\frac{e^{iz}}{z}\right| = \frac{e^{-y}}{|z|} \le \frac{e^{-y}}{R}.$

$$\int_{\gamma_{4a}} |f(z)||dz| \le \frac{1}{R}\int_0^R e^{-y}\,dy \le \frac{1}{R} \to 0$$

$$\int_{\gamma_{4c}} |f(z)||dz| \le \frac{1}{R} \to 0.$$

$$\int_{\gamma_{4b}} |f(z)||dz| \le e^{-R}\cdot 2R. \to 0$$

✓

$$\int_{\gamma_2}\frac{e^{iz}}{z}\,dz = \int_{\gamma_2}\frac{1}{z} + \underbrace{\text{analytic near } 0}_{|\,|\le M}\,dz$$

$$= \int_{\gamma_2}\frac{1}{z}\,dz \pm \underset{\substack{\downarrow \text{ as } \varepsilon\to 0\\ 0}}{M\cdot\pi\varepsilon}$$

$$= -\pi i \pm M\pi\varepsilon$$

so $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = -\int_{\gamma_2} f(z)\,dz - \int_{\gamma_4} f(z)\,dz$

$\lim_{R\to\infty}$! $\int_{-\infty}^{\varepsilon} f(z)\,dz + \int_{\varepsilon}^{\infty} f(z)\,dz = \pi i \pm M\pi\varepsilon$

$\lim_{\varepsilon\to 0}$! $\text{PV}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = \pi i$

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\pi}$$