Math 4200
Friday Nov 4

HW for Fri 11/11

§4.3 1, 2, 3, 9, 10, 14, 17, 20b
§4.4 1, 2, 3, 4, 5, 8, 9*

(I don't know how to do #9 – consider it a challenge)

Some of my favorite contour integrals:

1. $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where e.g. f is a rational fcn – it could be any fcn analytic in each argument.

this is a disguised contour integral around the unit circle:

if $\boxed{z = e^{i\theta}}$ $\Rightarrow \boxed{\frac{1}{z} = e^{-i\theta}}$

$dz = i e^{i\theta} d\theta$ so $d\theta = \frac{dz}{ie^{i\theta}} \doteq \boxed{-i\frac{dz}{z} = d\theta}$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\boxed{\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)}$ $\boxed{\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)}$

e.g: one you know:

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \oint_{|z|=1} \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 \frac{-i\,dz}{z} = \oint_{|z|=1} \frac{1}{4}\left(z + \frac{2}{z} + \frac{1}{z^2}\right](-i\,dz)$$

$$= 2\pi i\left[\operatorname{Res}(f, 0)\right]$$
$$= 2\pi i\left[-\frac{i}{2}\right] = \pi \ \checkmark$$

one you don't:

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta}, \quad \begin{matrix} a \neq \pm 1 \\ \text{real} \end{matrix} \quad (\text{so denom} \neq 0 \text{ for any } \theta)$$

$$\text{denom} = (1 - a\cos\theta)^2 + a^2(\sin^2\theta)$$

$$= \oint_{|z|=1} \frac{1}{1 + a^2 - a\left(z + \frac{1}{z}\right)} \frac{-i\,dz}{z}$$

$$= \oint_{|z|=1} \frac{-i}{-az^2 + (1+a^2)z - a}\, dz$$

$$-a\left(z^2 - \left(\tfrac{1}{a} + a\right)z + 1\right) = -a\left(z - \tfrac{1}{a}\right)(z - a)$$

$$= \frac{i}{a}\oint_{|z|=1} \frac{1}{\left(z - \frac{1}{a}\right)(z - a)}\, dz$$

$|a| < 1 \Rightarrow \operatorname{Res}(f, a) = \frac{1}{a - \frac{1}{a}}$

$\Rightarrow \text{ans} = \frac{i}{a}\, 2\pi i\, \frac{1}{a - \frac{1}{a}} = \frac{2\pi}{1 - a^2}$

$|a| > 1 \Rightarrow \operatorname{Res}\left(f, \frac{1}{a}\right) = \frac{1}{\frac{1}{a} - a}$

$\Rightarrow \text{ans} = \frac{i}{a}\, 2\pi i\, \frac{1}{\frac{1}{a} - a} = \frac{2\pi}{a^2 - 1}$

$\text{ans} = \frac{2\pi}{|a^2 - 1|}$

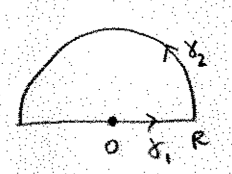
when $f = \frac{g}{h}$ and h is a factored polynomial, with a root of order k at z_0 (& $g(z_0) \neq 0$), then easiest way to get residue is from the coeff of $(z - z_0)^{k-1}$ in Taylor expansion of $(z - z_0)^k f(z)$!

(2) This sort:

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx \qquad , a \neq 0, a \text{ real.}$$

assume $a > 0$ $\quad [\cos(-ax) = \cos ax]$

first try (we've done this sort before): $f(z) = \dfrac{\cos az}{1+z^2} = \dfrac{\cos az}{(z-i)(z+i)}$



consider $\gamma = \gamma_1 + \gamma_2$, let $R \to \infty$

$$2\pi i \operatorname{Res}(f,i) = \int_\gamma f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\frown} f(z)\,dz$$

$$= 2\pi i \left[\frac{\cos ai}{2i}\right] = \pi \cosh a$$

$\int_{-R}^{R} f(x)\,dx \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx$

$\int_{\frown} f(z)\,dz \xrightarrow{\text{as } R\to\infty ?} 0$ NO

$$\frac{\cos az}{1+z^2} = \frac{1}{2}\left[e^{ia(x+iy)} + e^{-ia(x+iy)}\right]$$

$$= \frac{1}{1+z^2}\,\frac{1}{2}\left[e^{iax-ay} + e^{-iax+ay}\right]$$

trouble in lower half plane

trouble in upper half plane

How to fix:

look above.

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,dx.$$

Let $f(z) = \dfrac{e^{iaz}}{1+z^2}$ $\qquad$ then on γ_2, $|f(z)| \le \dfrac{1}{|z|^2-1} = \dfrac{1}{R^2-1}$

So $\int_{\gamma_2} |f(z)||dz| \le \dfrac{\pi R}{R^2-1} \to 0$ as $R \to \infty$.

So

$$2\pi i \operatorname{Res}(f,i) = \int_\gamma f(z)\,dz = \int_{-R}^{R} \frac{e^{iax}}{1+x^2}\,dx + \int_{\gamma_2} f(z)\,dz$$

$$= 2\pi i \left[\frac{e^{iai}}{i+i}\right]$$

$$\boxed{= \pi e^{-a}}$$

$\int_{-R}^{R} \frac{e^{iax}}{1+x^2}\,dx \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,dx$

$\int_{\gamma_2} f(z)\,dz \to 0$ as $R \to \infty$

take Real parts

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx = \frac{\pi}{e^a}, \quad a > 0$$

This is really my favorite

(3) We studied $\pi \cot \pi z$ – it has a simple pole at each integer, with residue 1.

Let's consider $\frac{1}{z^2} \pi \cot \pi z$. for $n \neq 0$ $\mathrm{Res}(f, n) = \frac{1}{n^2}$

for $n=0$ pole of order 3, need Laurent expansion which we can deduce from the Taylor series for $\pi z \cot \pi z$ centered at 0

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> taylor(Pi*z*cot(Pi*z),z=0,10);
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$$1-\frac{1}{3}\pi^2 z^2-\frac{1}{45}\pi^4 z^4-\frac{2}{945}\pi^6 z^6-\frac{1}{4725}\pi^8 z^8+O(z^{10})$$

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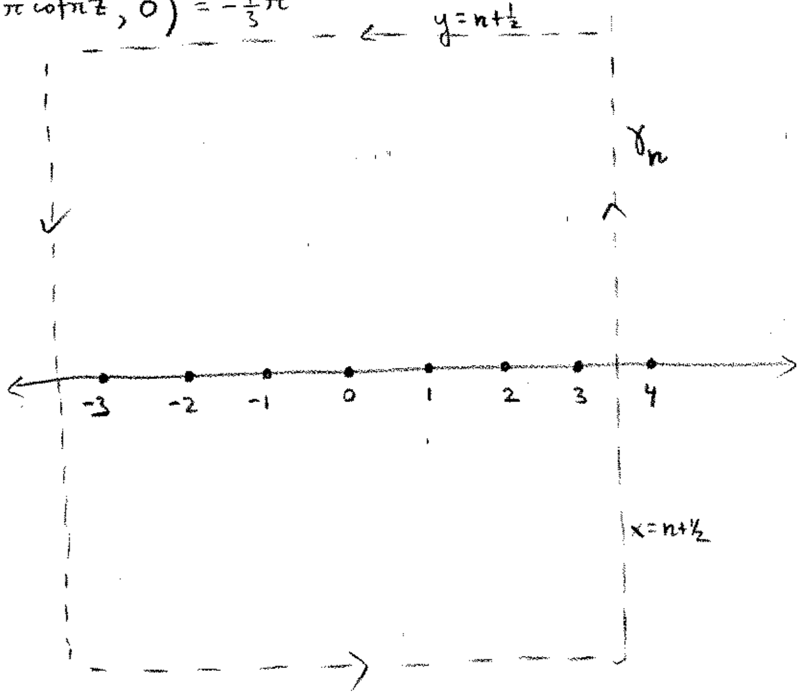
$\Rightarrow \mathrm{res}\left(\frac{1}{z^2}\pi\cot\pi z, 0\right) = -\frac{1}{3}\pi^2$

Now:



γ_n traces a square thru $x=\pm(n+\frac{1}{2})$
$y=\pm(n+\frac{1}{2})$

$$\frac{\cos \pi z}{\sin \pi z} = \frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\, i$$

$$= i\left[\frac{1+e^{-2i\pi z}}{1-e^{-2i\pi z}}\right]$$

$$= i\left[\frac{1+e^{-2i\pi x}\, e^{2\pi y}}{1-e^{-2i\pi x}\, e^{2\pi y}}\right] \quad *$$

along the vertical segments of γ_n

$$e^{-2i\pi(n+\frac{1}{2})} = e^{-i\pi} = -1$$
$$e^{-2i\pi(-)(n+\frac{1}{2})} = e^{i\pi} = -1$$

so $$\frac{\cos\pi z}{\sin\pi z} = i\left[\frac{1-e^{2\pi y}}{1+e^{2\pi y}}\right]$$

$$\left|\frac{\cos\pi z}{\sin\pi z}\right| \leq 1 \quad \text{on vertical sides.}$$

check top & bottom separately, use * on page 3!

on horizontal sides, n large

$$\left|\frac{\cos\pi z}{\sin\pi z}\right| \leq \frac{e^{2\pi(n+\frac{1}{2})}+1}{e^{2\pi(n+\frac{1}{2})}-1} = \frac{1+e^{-2\pi(n+\frac{1}{2})}}{1-e^{-2\pi(n+\frac{1}{2})}} < 2 \quad n \text{ large.}$$

$$\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$$

$$\Rightarrow \left|\oint_{\gamma_n} \frac{1}{z^2}\pi\cot\pi z\,dz\right| \leq 4\cdot 2(n+\tfrac{1}{2})\cdot\frac{1}{(n+\frac{1}{2})^2}\cdot 2 \to 0 \quad \text{as } n\to\infty$$

The contour integral disappears!

$$\text{But } \int_{\gamma_n} \frac{1}{z^2}\pi\cot\pi z\,dz = 2\pi i\sum_{j=-n}^{n}\operatorname{res}(f,j)$$
$$= 2\pi i\cdot\left[2\sum_{j=1}^{n}\frac{1}{j^2} + \underbrace{\operatorname{Res}(f,0)}_{-\frac{1}{3}\pi^2}\right]$$

$$\Big\downarrow\, n\to\infty$$

$$0$$

$$\boxed{\Rightarrow \sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{1}{6}\pi^2.}$$

What is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$?

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^6}$? You can read these off page 3!

How about $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$? This is §4.4 #9.