Math 4200

Wed Nov 30

Chapter 5: Conformal maps

This is actually an in depth return to the ideas we began the course with:

Recall the chain rule for curves:

f analytic at z_0

$\gamma: I \to \mathbb{C}$ diff'ble, $\gamma(t_0) = z_0$

$\Rightarrow (f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$

so $f'(z_0) \neq 0 \Rightarrow$ tangent vectors at z_0 are mapped to tangent vectors at $f(z_0)$, all of which are scaled by $|f'(z_0)|$ and rotated by $\arg(f'(z_0))$.

We call(ed) a map conformal iff it has this infinitesimal (i.e. on tangent vectors) rotation and scaling property $\forall z \in A$ (the domain). By Cauchy-Riemann this is equivalent to
$f: A \to \mathbb{C}$ analytic and $f'(z) \neq 0$ $\forall z \in A$

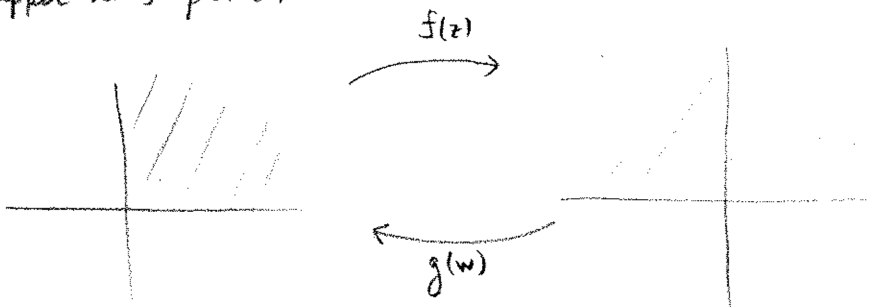
In chapter 5 we are interested in finding bijective conformal maps between open subsets A, B of $\mathbb{C}$ (Then A and B are called conformally equivalent).

(Applications include PDE's & geometry)

Of course we remember many examples from chapter 1...

example:

how many conformal bijections can you find between the 1st quadrant and the upper half plane?

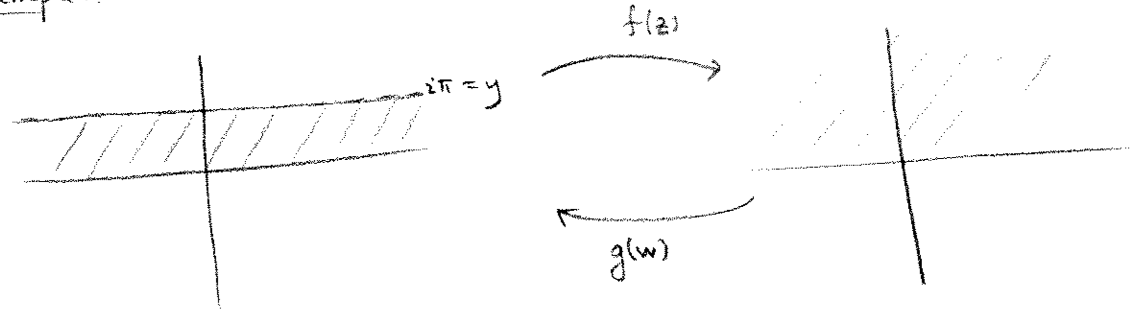


$f(z) = z^2$

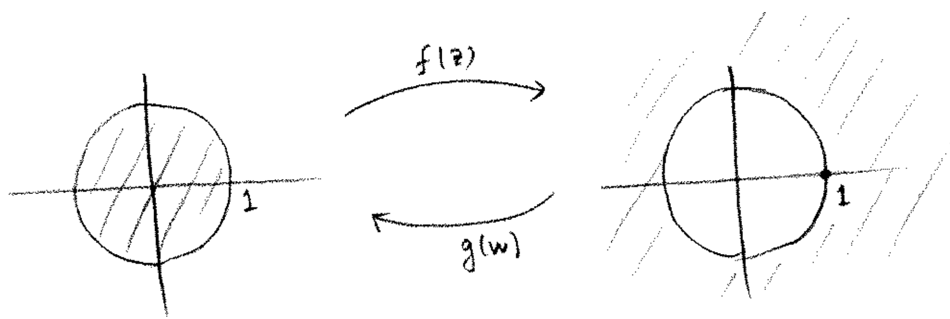
$f(z) = az^2 + b$ $a > 0$, b real

is this all?

examples.



$f(z) = ae^z + b$ $\quad a>0,\ b \in \mathbb{R}$ $\quad$ is this all?



$f(z) = e^{i\theta}\frac{1}{z}$ $\quad$ is this all?

note $f(0) = \infty$, so this map should really be considered on the Riemann sphere.

Riemann Mapping Theorem (version 1)

Let $A \neq \mathbb{C}$ be open and simply connected, $z_0 \in A$

Then $\exists !\ f: A \to D(0,1)$ s.t. f is a conformal bijection, $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}$, (arg = 0) $f'(z_0) > 0$

3 real degrees of freedom!

proof of existence is hard - consult more advanced text

proof of uniqueness is "easy":

Suppose $\exists\ f_1, f_2$ as above. consider

$g(z) = f_2 \circ f_1^{-1} : D \to D.$

then $g(0) = 0$, so $G(z) = \begin{cases} \frac{g(z)}{z} & z \neq 0 \\ g'(0) & z = 0 \end{cases}$ is analytic.

$|G(z)| \leq \frac{1}{r}$ on $|z| = r \Rightarrow$ for $|z| \leq r$, by max mod. principle

as $r \to 1 \Rightarrow |G(z)| \leq 1$ in $D(0,1)$

$\Rightarrow |g(z)| \leq |z|$ in $D(0,1)$

[we are reproducing Schwarz Lemma]

But by symmetric argument, $g^{-1}(z) = f_1 \circ f_2^{-1}$

also satisfies $|g^{-1}(z)| \leq |z| \quad \forall z \in D(0,1)$

$\Rightarrow |g^{-1}(g(z))| \leq |g(z)|$

$|z| \leq |g(z)|$

$\Rightarrow |g(z)| = |z|$

$\Rightarrow |G(z)| \equiv 1$

$\Rightarrow G(z) = \text{const} = e^{i\theta}$

$\Rightarrow g(z) = e^{i\theta} z$

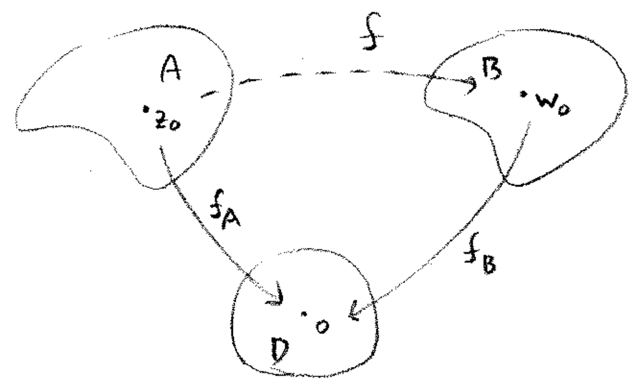
but $g'(0) = (f_2 \circ f_1^{-1})'(0)$

$= f_2'(z_0)\underbrace{(f_1^{-1})'(0)}_{\frac{1}{f_1'(z_0)}}$ is real positive # $\Rightarrow g(z) = z$ $\Rightarrow f_2 = f_1$!

Riemann Mapping Theorem (version 2)

Let $A, B \subset \mathbb{C}$ be simply connected and not all of $\mathbb{C}$.
Let $z_0 \in A$, $w_0 \in B$
Then $\exists!\ f : A \to B$ conformal bijection s.t. $f(z_0) = w_0$
$\arg f'(z_0) > 0$

pf:



$\exists\ f_A, f_B$ by RMT$_1$
Let $f := f_B^{-1} \circ f_A$
$f'(z_0) = (f_B^{-1})'(0)\, f_A'(z_0) \in \mathbb{R}^+$
If $\exists\ f_2$ with identical props as f
then $f_B \circ f_2 \circ f_A^{-1}$ maps disk to itself,
$0 \to 0$
deriv$(0) \in \mathbb{R}^+$
$\Rightarrow f_B \circ f_2 \circ f_A^{-1}(z) = z$
as on page 2.

∎

So did we get all the conformal maps in our 3 examples?

ans: not in any of them!

We will be helped by studying
fractional linear transformations (FLT's)

$$\boxed{f(z) = \frac{az+b}{cz+d}} \qquad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc \neq 0.$$

(else f is constant!)

these functions are meromorphic, with at most a single simple pole $(c \neq 0)$, and in fact are bijections from the Riemann sphere to itself. This will follow from a few simple computations...

example $f(z) = az + b$, $a \neq 0$ is an affine map from $\mathbb{C}$ to $\mathbb{C}$. These are the only 1-1 conformal maps defined on $\mathbb{C}$! (in HW).

Note, why is there no conformal bijection $f : \mathbb{C} \to D(0,1)$?

ans = Liouville.

Properties of FLT's.

① If $f(z) := \frac{az+b}{cz+d}$, $g(w) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Then $g(f(z)) = \frac{Az+B}{Cz+D}$

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

defined up to non-zero scalar mults

"The" matrix of the composition is the product of the matrices!

check it:

② If $f(z) = \frac{az+b}{cz+d}$ then $f^{-1}(w) = \frac{dw-b}{-cw+a}$ with det $\neq 0$

pf: adjoint formula for inverse matrix!

example $f(z) = \frac{z+2}{z-1}$ $\quad f^{-1}(z) = \frac{-z-2}{-z+1}$

$$f\circ f^{-1}(z) = \frac{\frac{-z-2}{-z+1}+2}{\frac{-z-2}{-z+1}-1} = \frac{-z-2+2(-z+1)}{-z-2-(-z+1)} = \frac{-3z}{-3} = z \checkmark$$

Cor Every LFT is conformal and 1-1 on its domain.

pf $f^{-1}\circ f(z) = z \Rightarrow$ 1-1;

$\Rightarrow (f^{-1})'(w)\cdot f'(z) = 1 \Rightarrow f'(z) \neq 0$ (could check directly too)

We can also see from this that LFT's extend to be bijections of the Riemann sphere.

③ FLT's take circles and lines to circles or lines.

pf: True for $T_1(z) = z+a$ (translation)

$T_2(z) = cz$ (scaling and rotation)

$*$ $T_3(z) = \frac{1}{z}$ (inversion)

this one needs to be checked! (tricky but short, see page 330)

and every FLT is a composition of at most 4 T_j's (see HW).

example: $f(z) = \frac{z+i}{2z+3}$

$\frac{1}{2} = \frac{z+3/2}{2z+3}$ $\quad$ so $f(z) = \frac{1}{2} + (i-3/2)\frac{1}{2z+3}$

$f_1(z) = 2z$

$f_2(w) = w+3$

$f_3(3) = \frac{1}{3}$

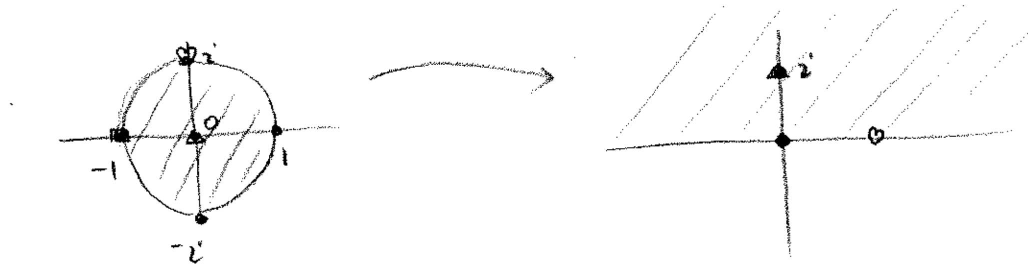
$f_4(w) = (i-3/2)w$

$f_5(\xi) = \frac{1}{2}+\xi$

$f = f_5\circ f_4\circ f_3\circ f_2\circ f_1$

<u>example</u>

$$f(z) = \left(\frac{z-1}{z+1}\right)\overset{-i}{\overset{\|}{\frac{i+1}{i-1}}}$$



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To send

$a \to 0$
$b \to \infty$
$c \to d$

use $f(z) = \left(\frac{z-a}{z-b}\right)\left(\frac{c-b}{c-a}\right) d$

Composing such $f(z)$ and their inverses you can find a LFT which transforms any 3 distinct pts to any (other) 3 distinct pts.