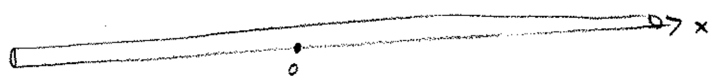
Math 4200
Monday Nov. 28

Fourier transform & 1-d heat eqtn

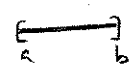
- Fourier Transform : The real place to study this is in a PDE course (e.g. 5000-level), but we will do one example carefully, namely the solution to the IVP for the heat equation in all of space. We will deal with $n=1$ space dimension, although these techniques readily adapt to more space dims.

Heat eqtn:

$n=1$
(don't need div thm in this case)



$u(x,t)$ = temp at position x, time t
consider any fixed interval $I=[a,b] \subset \mathbb{R}$

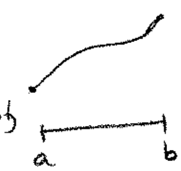


assumption: heat flows from hot to cold, at rate prop to heat gradient.
no heat is added or subtracted to the system

i.e.

$$\frac{d}{dt}\underbrace{\int_a^b c\rho u(x,t)\,dx}_{\substack{\text{total heat in interval} \\ \text{(energy)}, \ c = \text{specific ht}, \ \rho = \text{dens}}} = \kappa\left[u_x(b) - u_x(a)\right]$$

κ = conductivity



$$\Rightarrow \int_a^b c\rho u_t(x,t)\,dx = \kappa \int_a^b u_{xx}\,dx$$

$$\Rightarrow \int_a^b (c\rho u_t - \kappa u_{xx})\,dx = 0 \quad \forall [a,b]$$

$$\Rightarrow u_t = \frac{\kappa}{c\rho} u_{xx} \qquad \text{heat eqtn}$$

you can rescale x or t to get normalized heat eqtn
(e.g. $v(x,t) = u(x, \frac{\kappa}{c\rho} t)$)

$$\boxed{u_t = u_{xx}}$$

$\Delta u = \operatorname{div}(\nabla u)$

Recall
n space variables:

$$\frac{d}{dt}\iint_\Omega c\rho u(x,t)\,d\vec{x} = \int_{\partial\Omega} \kappa(\nabla u \cdot \vec{n})\,dA = \iint_\Omega \kappa(\Delta u)\,d\vec{x}$$

test region Ω.

$$\Rightarrow u_t = \frac{\kappa}{c\rho}\Delta u, \quad \text{normalized to } \boxed{u_t = \Delta u}$$

heat equation = diffusion equation, lots of applications.
"Gaussian blur" in computer graphics...

Fourier Transform

$f : \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R} \to \mathbb{R}$)
$f(x)$

$$\hat{f}(\omega) := (\mathcal{F} f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

Fourier transform of f.

$g : \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R} \to \mathbb{R}$)
$g(\omega)$

$$\check{g}(x) := (\mathcal{F}^{-1} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x}\, d\omega$$

inverse Fourier transform of g.
[it is some work to show this is the inverse of Fourier transform, on a suitable class of functions].

Properties of Fourier Transform

(1) $\mathcal{F}(c_1 f + c_2 g) = c_1 \mathcal{F}(f) + c_2 \mathcal{F}(g)$ (linearity).

(2) $(\mathcal{F} f')(\omega) = i\omega \hat{f}(\omega)$ ← derivatives are converted to multiplication. This is what made Laplace transform and Fourier series useful in ODE's

check:
$$\int_{-\infty}^{\infty} f'(x) e^{-i\omega x}\, dx = \underbrace{f(x) e^{-i\omega x}\Big]_{-\infty}^{\infty}}_{\substack{=0 \text{ if} \\ f\to 0 \text{ at } \infty}} - \underbrace{\int_{-\infty}^{\infty} f(x)(-i\omega) e^{-i\omega x}\, dx}_{i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx} \quad \blacksquare$$

We will need another property (convolution) soon, but using (2) we can already get pretty far in trying to solve the IVP for heat eqtn!

IVP
$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & \text{initial temp distribution, (assumed to decay to zero at } \infty) \end{cases}$$

$$\Rightarrow \mathcal{F}(u_t)(\omega) = \mathcal{F}(u_{xx})(\omega)$$

Fourier transform in x-variable, for each fixed t.

$$\mathcal{F}(u_t)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t) e^{-ix\omega}\, dx = \frac{d}{dt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ix\omega}\, dx = \frac{d}{dt} \hat{u}(\omega, t)$$

$$\mathcal{F}(u_{xx})(\omega) = (i\omega)(i\omega) \mathcal{F}(u(\cdot,t))(\omega) = -\omega^2 \hat{u}(\omega, t)$$

so, for each fixed ω, $\hat{u}(\omega,t)$ solves ODE:

IVP
$$\begin{cases} \frac{d}{dt} \hat{u}(\omega,t) = -\omega^2 \hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\boxed{\Rightarrow \hat{u}(\omega,t) = \hat{f}(\omega) e^{-\omega^2 t}}$$

Done?

We've solved our problem in Fourier transform land,
we just need to transform back to the real world.
In order to do so, we use convolution on the real line, which is almost identical to the convolution we did on $[0,2\pi]$ for the Poisson integral formula.

(3) Let $f, h : \mathbb{R} \to \mathbb{R}$.

$$(f*h)(x) := \int_{-\infty}^{\infty} f(x-y)h(y)\,dy = \int_{-\infty}^{\infty} f(y)h(x-y)\,dy = (h*f)(x)$$

"the convolution of f with h"

Then $\widehat{(f*h)}(\omega) = \hat{f}(\omega)\hat{h}(\omega)\sqrt{2\pi}$

check: $\widehat{(f*h)}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-y)h(y)\,dy \right] e^{-i\omega x}\,dx$

(Fubini) $$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Bigg(\underbrace{\int_{-\infty}^{\infty} f(x-y)\,h(y)\,e^{-i\omega x}\,dx}_{\int \left[\int_{-\infty}^{\infty} \left[f(x-y)e^{-i\omega(x-y)}\, h(y)e^{-i\omega y} \right] dx \right] dy} \Bigg) dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[h(y)\, e^{-i\omega y} \underbrace{\int_{-\infty}^{\infty} f(x-y)e^{-i\omega(x-y)}\,dx}_{\substack{\tilde{x} = x-y \\ d\tilde{x} = dx \\ \int_{-\infty}^{\infty} f(\tilde{x})e^{-i\omega\tilde{x}}\,d\tilde{x}}} \Big] dy$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \right) \left[\int_{-\infty}^{\infty} h(y)e^{-i\omega y}\,dy \right]$$

∎

So, we can invert the box on page 2:

$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-\omega^2 t}$

$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \left[f * \hat{f}^{-1}(e^{-\omega^2 t}) \right]$

here's where we use a contour integral!

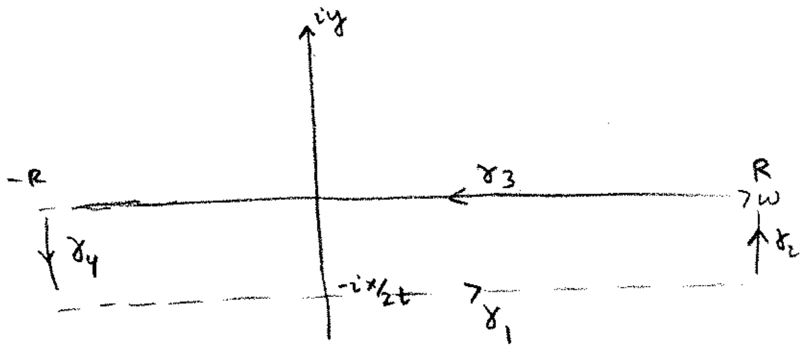
$$\mathcal{F}^{-1}\left(e^{-w^2 t}\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2 t} e^{iwx}\, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\left(w^2 - \frac{iwx}{t}\right)}\, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\left(w - \frac{ix}{2t}\right)^2} e^{-\frac{x^2}{4t}}\, dw$$

$$= e^{-\frac{x^2}{4t}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\left(w - \frac{ix}{2t}\right)^2}\, dw}_{*}$$

in the $z = w + iy$ plane consider $\gamma = \{ y = -\frac{x}{2t}$



$$* = \lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz \qquad f(z) = e^{-tz^2}$$

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\,dz = 0.$$

$$\left| \int_{\gamma_2} f(z)\,dz \right| \le \left|\frac{x}{2t}\right| \max_{\gamma_2} \left| e^{-tz^2} \right|$$

$$\left| e^{-t(R+iy)^2} \right|$$

$$\left| \int_{\gamma_4} f(z)\,dz \right| \to 0 \text{ too} \qquad e^{-t(R^2+y^2)} \to 0 \text{ as } R\to\infty$$

$$\Rightarrow * = \int_{-\infty}^{\infty} e^{-tw^2}\, dw \quad \left(= \lim_{R\to\infty} - \int_{\gamma_3} f(z)\,dz \right)_*$$

You did this integral in 2210 using polar coords?

$$\left(\int_{-\infty}^{\infty} e^{-tx^2} dx \right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-tx^2} e^{-ty^2}\, dA = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-tr^2}\, r\, dr\, d\theta = \left.\frac{e^{-tr^2}}{-2t}\right]_0^{\infty} \cdot 2\pi$$

$$= \frac{\pi}{t}$$

deduce

$$\tilde{f}^{-1}\left(e^{-w^2 t}\right)(x) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{t}}\, e^{-\frac{x^2}{4t}} = \frac{1}{\sqrt{2t}}\, e^{-\frac{x^2}{4t}}$$

so

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\, f *$$

(convolution commutes)

$$\boxed{u(x,t) = \underbrace{\int_{-\infty}^{\infty} f(x-y)\,\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{y^2}{4t}}\,dy}_{} = \int_{-\infty}^{\infty} f(y)\,\underbrace{\frac{1}{\sqrt{4\pi t}}\, e^{-(x-y)^2/4t}}_{}\,dy}$$

Solves IVP for heat eqtn

this expression shows how the solution $u(x,t)$ averages the initial data "f".

this function of (x,t) & solves the Heat eqtn & represents the soltn when a unit heat source is placed at y, at time $t=0$.

this is an integral superposition to the linear PDE $u_t - u_{xx} = 0$ which solves the IVP

Approximate identity properties: (as $t \searrow 0^+$)

$\tilde{y} = \frac{y}{2\sqrt{t}}$; $d\tilde{y} = \frac{1}{2\sqrt{t}}\,dy$

- for $t>0$, $\displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\, e^{-y^2/4t}\,dy = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{t}\sqrt{\pi}}\, e^{-\tilde{y}^2}\, 2\sqrt{t}\,d\tilde{y} = 1$

- for $t>0$, $\displaystyle\int_{|y|<\delta} \frac{1}{\sqrt{4\pi t}}\, e^{-y^2/4t}\,dy = \int_{|\tilde{y}|<\frac{\delta}{2\sqrt{t}}} \frac{1}{\sqrt{\pi}}\, e^{-\tilde{y}^2}\,d\tilde{y} \longrightarrow 1$ as $t \searrow 0^+$

$\boxed{\text{Class Exercise IV:}}$ Assume $f(y)$ is <u>bounded</u> and <u>uniformly continuous</u> on $\mathbb{R}$

$\exists\, M<\infty$ $|f(y)| \le M$ $\forall y$

$\forall \varepsilon>0\ \exists\ \delta>0$ s.t. $|y_1-y_2|<\delta \Rightarrow |f(y_1)-f(y_2)|<\varepsilon$

Define $\tilde{u}(x,t) = \begin{cases} u(x,t) & t>0 \\ f(x) & t=0 \end{cases}$

Prove $\tilde{u}$ is continuous on $\{(x,t) \text{ s.t. } x\in\mathbb{R},\ t\ge 0\}$

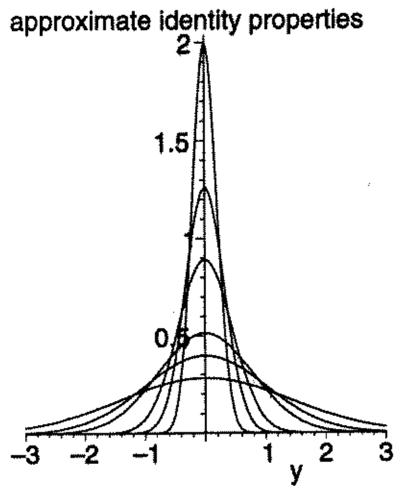
You may assume u is continuous for $t>0$ (in fact, it's C^∞, as one could justify from the boxed convolution formulas.

Thus the issue is to show $\tilde{u}(x,t)$ is continuous at all points $(x,0)$.

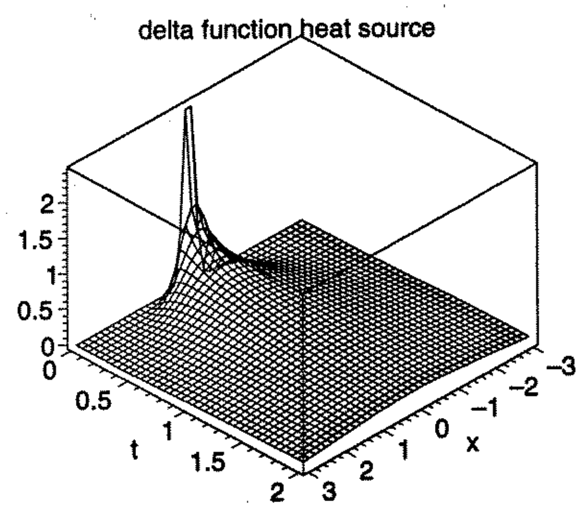
Use the approximate identity properties and ideas like those in Monday 11/21 notes

Math 4200
Fourier Transform
November 28, 2005

```
> with(plots):
> g:=(y,t)->1/sqrt(4*Pi*t)*exp(-y^2/(4*t)):
> plot({g(y,1),g(y,.5),g(y,.3),g(y,.1),g(y,.05),g(y,.02)},
  y=-3..3,color=black,title=`approximate identity properties`);
```




```
> plot3d(g(x,t),x=-3..3,t=0.01..2,color=black,axes=boxed,style=wiref
  rame,grid=[40,40],title=`delta function heat source`);
```




```
> simplify(diff(g(x,t),t)-diff(g(x,t),x,x));
  #the integral formula represents an integral
  #superposition of solutions to the heat equation:
```

$$0$$

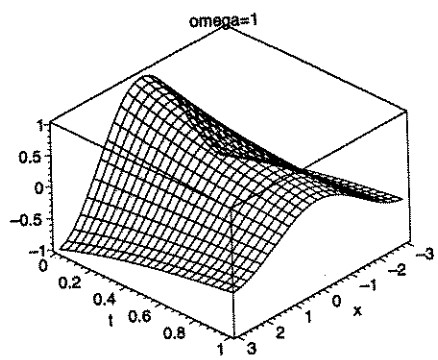
<u>Remark</u>: Although the solution to the Heat equation as been understood since the 1800's, a much more difficult (actually impossible) problem is to solve the heat equation in the −time direction. This is what you want to approximate in <u>image processing</u>, since a lot of image blurring is due to random diffusion of light.

Mathematically, going in the negative time direction isn't possible unless you control (or eliminate) large ω: this is because separated solution to the heat equation are, for example,

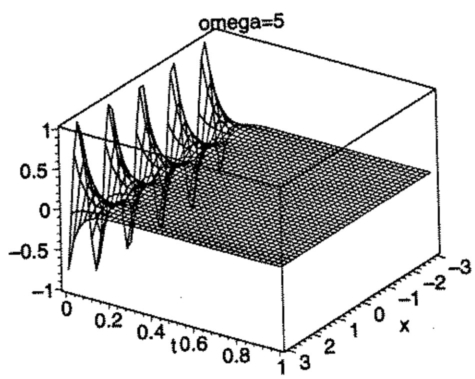
$$(\cos \omega x)\, e^{-\omega^2 t}$$

and if ω is large then these solutions essentially disappear in a short time!

```
> plot3d(cos(x)*exp(-t),x=-3..3,
  t=0..1,style=wireframe,color=black,axes=boxed,
  title='omega=1');
```




```
> plot3d(cos(5*x)*exp(-25*t),x=-3..3,
  t=0..1,style=wireframe,grid=[40,40],
  color=black,axes=boxed,
  title='omega=5');
```




```
>
```