Math 4200

Maple play, November 21

Work for the Poisson integral formula for harmonic functions on the unit disk

```
> with(plots):
> g:=(x,y)->(1-x^2-y^2)/(1-2*x*cos(theta)-2*y*sin(theta)
  +x^2+y^2);
```

$$g := (x, y) \to \frac{1 - x^2 - y^2}{1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2}$$

```
> diff(g(x,y),x,x)+diff(g(x,y),y,y);
```

$$-\frac{4}{1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2} + \frac{4\,x\,(-2\cos(\theta) + 2\,x)}{(1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2)^2}$$
$$+ \frac{2\,(1 - x^2 - y^2)\,(-2\cos(\theta) + 2\,x)^2}{(1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2)^3} - \frac{4\,(1 - x^2 - y^2)}{(1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2)^2}$$
$$+ \frac{4\,y\,(-2\sin(\theta) + 2\,y)}{(1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2)^2} + \frac{2\,(1 - x^2 - y^2)\,(-2\sin(\theta) + 2\,y)^2}{(1 - 2\,x\cos(\theta) - 2\,y\sin(\theta) + x^2 + y^2)^3}$$

```
> simplify(%); #gettin zero here shows the sense in which
  #the Poisson integral formula represents
  #a harmonic function as an integral superposition
  #of harmonic functions.
```

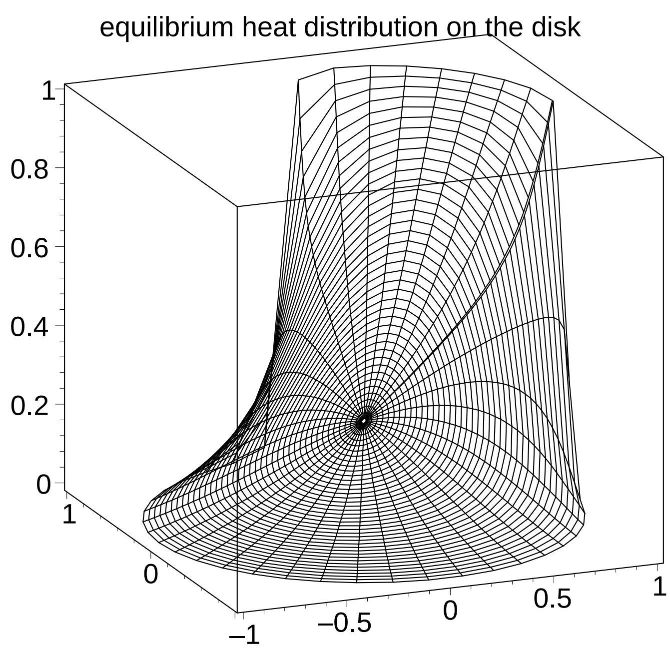
$$0$$

Here's an example where the boundary values are 1 on a quarter circle, and zero on the rest of the circle: (Could you write this function using the imaginary part of some complex logarithm expression?)

```
> u:=(rho,phi)->1/(2*Pi)*evalf(int((1-rho^2)/(1-2*rho*cos(theta-phi)
  +rho^2),theta=0..Pi/2));
```

$$u := (\rho, \phi) \to \frac{1}{2}\,\frac{\mathrm{evalf}\left(\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\rho^2}{1-2\,\rho\cos(\theta-\phi)+\rho^2}\,d\theta\right)}{\pi}$$

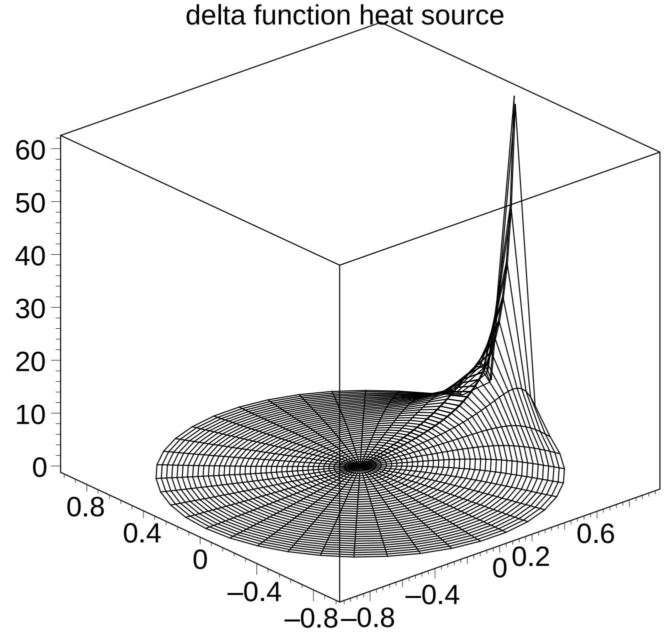
```
> plot3d([rho*cos(phi),rho*sin(phi),u(rho,phi)],rho=0.01..(.99),phi=
  0..2.05*Pi,grid=[40,40],axes=boxed,style=wireframe,
  color=black,title=‘equilibrium heat distribution on the disk‘);
```



So, without any work, what is the value of this harmonic function at the origin - precisely?

Understanding the Poison integral as an integral superposition of source functions:

```
> plot3d([rho*cos(phi),rho*sin(phi),(1/2*Pi)*
  (1-rho^2)/(1-2*rho*cos(phi)+rho^2)],rho=0..(.95),
  phi=0..2.05*Pi,grid=[40,40],axes=boxed,style=wireframe,
  color=black,title=‘delta function heat source‘);
```

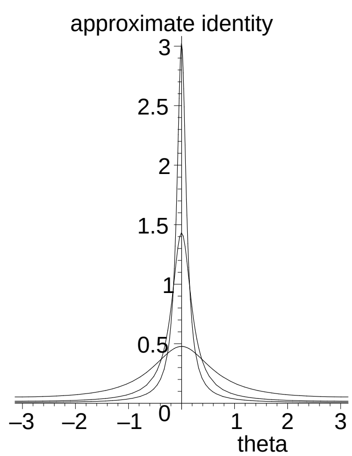


Illustrating the approximate identity nature of the function g:

```
> g:=(rho,theta)->1/(2*Pi)*(1-rho^2)/(1-2*rho*cos(theta)+rho^2);
```

$$g := (\rho, \theta) \to \frac{1}{2} \frac{1-\rho^2}{\pi (1 - 2 \rho \cos(\theta) + \rho^2)}$$

```
> plot({g(.5,theta),g(.8,theta),g(.9,theta)},theta=-Pi..Pi,
  color=black,title=‘approximate identity‘);
```



[>