Math 4200
Monday Nov 21

- Heat eqtn (& Laplace eqtn) derivation on page 5 of Friday's notes

- Poisson integral representation of harmonic functions in disks. (§2.5).
  (because $v(x,y) = u(ax,ay)$ is harmonic iff $u(x,y)$ is harmonic, it suffices to consider unit disks. Because $v(x,y) = u(x-h,y-k)$ is harmonic iff $u(x,y)$ is harmonic, it suffices to consider the unit disk centered at the origin)

<u>Derivation</u>: As we well know, if f is analytic in the unit disk, and $|z|<1$,

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

We wish to extract a formula for $u(z) = Re(f(z))$ in terms of just the boundary values of u. Unfortunately, you can't just take Re part of both sides above, because the Re & Im of $f(\zeta)$ will get jumbled together.

Power series trickery saves us once again:

$$\int_{|\zeta|=1} \frac{f(\zeta)}{1 - z/\zeta}\ \underbrace{\frac{1}{2\pi i}\frac{d\zeta}{\zeta}}_{\text{factored out}}$$

if $\zeta = e^{i\theta}$ $\quad \frac{d\zeta}{\zeta} = i\,d\theta$

so $\frac{1}{2\pi i}\frac{d\zeta}{\zeta} = \frac{1}{2\pi}d\theta$ is real

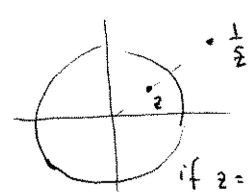
$$= \int_{|\zeta|=1} f(\zeta)\Big[\underbrace{1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots}_{\text{add conjugate to get something real here}}\Big]\frac{d\theta}{2\pi}$$

$$\int_{|\zeta|=1} f(\zeta)\Big[1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots + \underbrace{\frac{\bar z}{\bar\zeta} + \frac{\bar z^2}{\bar\zeta^2} + \cdots}_{= \bar z \zeta + (\bar z \zeta)^2 + \cdots}\Big]\frac{1}{2\pi i}\frac{d\zeta}{\zeta}$$

$$\frac{1}{1 - z/\zeta} + \frac{\bar z \zeta}{1 - \bar z \zeta}$$

yields

$$f(z) \overset{?}{=} \int_{|\zeta|=1} f(\zeta)\Big[\overbrace{\frac{1}{\zeta - z} + \frac{\bar z}{1 - \bar z \zeta}}^{\text{real!}}\Big]\frac{d\zeta}{2\pi i}$$



if $z = \rho e^{i\phi}$
$\frac{1}{\bar z} = \frac{1}{\rho}e^{i\phi}$

And this integral identity is true because $\frac{1}{1-\bar z \zeta}$ has a pole at $\frac{1}{\bar z}$, which is <u>outside</u> the unit circle

So
$$f(z) = \int_{|\zeta|=1} f(\zeta)\left[\frac{1}{1-z/\zeta} + \frac{\bar z \zeta}{1-\bar z\zeta}\right]\frac{d\zeta}{\zeta}\frac{1}{2\pi i}$$

Let $\zeta = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$
$\quad z = \rho e^{i\phi}$ $\quad \rho < 1$

$$(u+iv)(z) = \int_0^{2\pi} (u+iv)(\zeta)\left[\frac{1 - \cancel{\bar z\zeta} + \cancel{\bar z \zeta} - |z|^2}{(1-z\bar\zeta)(1-\bar z\zeta)}\right]\frac{d\theta}{2\pi}$$

$$\left[\frac{1-\rho^2}{1-2\rho\cos(\theta-\phi)+\rho^2}\right]\frac{d\theta}{2\pi}$$

$$1 - z\bar\zeta - \bar z\zeta + |z|^2$$
$$= 1 - \rho e^{-i\phi}e^{i\theta} - \rho e^{i\phi}e^{-i\theta} + \rho^2$$
$$= 1 - 2\rho\cos(\theta-\phi) + \rho^2$$

Taking real parts
yields Poisson integral formula:

$$\boxed{u(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\left(\frac{1-\rho^2}{1-2\rho\cos(\theta-\phi)+\rho^2}\right)d\theta}$$

this is a convolution integral on $[0,2\pi]$,

for $f(\theta) = u(e^{i\theta})$, $g(\theta) = \dfrac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}$

In general, for fcns f, g with domain $[0,2\pi]$ (& extended to be 2π periodic)

convolution $\quad (f * g)(\phi) := \displaystyle\int_0^{2\pi} f(\theta)\, g(\phi-\theta)\, d\theta$

$$= \int_0^{2\pi} g(\theta)\, f(\phi-\theta)\, d\theta = (g*f)(\phi)$$

Let $\tilde\theta = \phi - \theta$, $d\tilde\theta = -d\theta$
$(\ \theta = \phi - \tilde\theta)$

So 1st int $= -\displaystyle\int_{\phi}^{\phi-2\pi} f(\phi-\tilde\theta)\, g(\tilde\theta)\, d\tilde\theta$

two negatives; yields second integral.

So, we may also write

$$\boxed{u(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i(\phi-\theta)}\right)\left(\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}\right)d\theta}$$

You may have seen this convolution in 2280 when you were doing Fourier series: the n^{th} fourier coeffs of $f*g$ are the exact product of the n^{th} Fourier coef's of f with the n^{th} Fourier coeff of g.

There was a slightly different convolution operator for Laplace transform.

In fact, For any 2π-periodic, continuous (actually only need piecewise cont) for analogous result
function $f(\theta)$, if you define

THEOREM

$$u(\rho e^{i\phi}) := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\underbrace{\left(\frac{1-\rho^2}{1-2\rho\cos(\theta-\phi)+\rho^2}\right)}_{\frac{1-x^2-y^2}{1-2x\cos\theta-2y\sin\theta+x^2+y^2}} d\theta$$

$\left.\frac{1-x^2-y^2}{1-2x\cos\theta-2y\sin\theta+x^2+y^2}\right\}$ harmonic for $x^2+y^2<1$!

then this defines a harmonic function of $\rho e^{i\phi} = x+iy$ inside the unit disk (think of it as an integral superposition of harmonic fcns) and if $f(\theta)$ is continuous u extends to the closed disk, with boundary values $u(e^{i\phi}) = f(\phi)$.

i.e. you can always solve the Dirichlet problem for harmonic fcns in the unit disk.

"proof" of Theorem: as long as $x^2+y^2<1$, $k(x,y) = \dfrac{1-x^2-y^2}{1-2x\cos\theta-2y\sin\theta+x^2+y^2}$ is ∞'ly differentiable, and

$$\Delta k(x,y) = 0 \quad \text{(see Maple!)}$$

thus $\Delta u(x,y) = \Delta \int = \int \Delta_{(x,y)} = 0$

The proof that for f continuous u extends continuously to $|z|=1$, with $u(e^{i\theta}) = f(\theta)$ is an approximate identity argument.

Def $\{g_\rho(\theta)\}$ is an approximate identity on $[-\pi,\pi]$ (or $[0,2\pi]$)

if
- $g_\rho(\theta) \geq 0 \quad \forall \theta, \; 0<\rho<1$, ($g_\rho$ 2π periodic)
- $\int_{-\pi}^{\pi} g_\rho(\theta)\,d\theta = 1 \quad \forall\, 0<\rho<1$
- $\forall\, \varepsilon>0 \quad \lim_{\rho\to 1}\int_{-\varepsilon}^{\varepsilon} g_\rho(\theta)\,d\theta = 1$

(equivalent to $\lim_{\rho\to 1}\int_{\varepsilon}^{2\pi-\varepsilon} g_\rho(\theta)\,d\theta = 0$)

} $g_\rho \to$ "delta fcn" as $\rho\to 0$

Example $g_\rho(\theta) = \frac{1}{2\pi}\left(\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}\right)$ defines an approximate identy family

- $g_\rho(\theta) \geq 0 \quad \forall \theta, \; \forall 0<\rho<1$, since num >0 & denom $= (1-\rho\cos\theta)^2 + \rho^2\sin^2\theta > 0$.

- $\displaystyle\int_{-\pi}^{\pi} g_\rho(\theta)\,d\theta = 1 \quad \forall\ 0<\rho<1$

proof 1 : This was our first ~~proof~~ computation using the residue thm! (page 1 Nov 4)

$$\text{“} \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = \frac{2\pi}{|a^2-1|} \text{”}$$

proof 2 : $u \equiv 1$ is harmonic, $= \mathrm{Re}\,(1+0i)$,

so P.I.F. says

$$1 = u(\rho e^{i0}) = \frac{1}{2\pi}\int_0^{2\pi} 1 \left(\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}\right) d\theta$$

- $\displaystyle g_\rho(\theta) = \frac{1}{2\pi}\left(\frac{1-\rho^2}{(1-\rho)^2 + 2\rho(1-\cos\theta)}\right) = \frac{1}{2\pi}\left(\frac{1+\rho}{(1-\rho) + \frac{2\rho}{1-\rho}(1-\cos\theta)}\right)$

$$\begin{matrix}\cos\theta < 1-\tilde{\varepsilon} \\ 1-\cos\theta > \tilde{\varepsilon}\end{matrix} \Rightarrow g_\rho(\theta) < \frac{1}{2\pi}\left(\frac{1+\rho}{\frac{2\rho}{1-\rho}\varepsilon}\right) = \frac{1}{2\pi}\frac{(1+\rho)(1-\rho)}{2\rho\varepsilon} \longrightarrow 0 \text{ as } \rho\to 1$$

so $g_\rho(\theta) \to 0$ uniformly on $\varepsilon \le \theta \le 2\pi-\varepsilon$, proves 3rd bullet (see Maple pictures).

Continuity on closed disk & $u(e^{i\theta}) = f(\theta)$ follows from general approx identity (mollification) result:

Thm Let f cont & 2π period, $g_\rho(\theta)$ an approx identity.

Then $f * g_\rho(\phi) \to f(\phi)$ uniformly on $[0,2\pi]$, as $\rho\to 1$

proof :
$$\begin{aligned}(f*g_\rho)(\phi) - f(\phi) &= \int_0^{2\pi} f(\phi-\theta)\,g_\rho(\theta)\,d\theta - f(\phi) \\ &= \int_{-\pi}^{\pi} [f(\phi-\theta) - f(\phi)]\,g_\rho(\theta)\,d\theta \qquad \left(\text{since } \int g_\rho(\theta)\,d\theta = 1\right) \\ &= \int_{-\varepsilon}^{\varepsilon} [f(\phi-\theta)-f(\phi)]\,g_\rho(\theta)\,d\theta + \int_{\varepsilon}^{2\pi-\varepsilon} [f(\phi-\theta)-f(\phi)]\,g_\rho(\theta)\,d\theta \\ &= I_1 + I_2\end{aligned}$$

since f is uniformly cont on $[0,2\pi]$,

$$\omega(\varepsilon) := \max_{|\theta_1-\theta_2|\le\varepsilon} |f(\theta_1)-f(\theta_2)| \to 0 \text{ as } \varepsilon\to 0$$

so $\displaystyle |I_1| \le \int_{-\varepsilon}^{\varepsilon} (\omega(\varepsilon))\,g_\rho(\theta)\,d\theta \le \omega(\varepsilon)$, since $\displaystyle\int_0^{2\pi} g_\rho(\theta)\,d\theta = 1$

Let $\max|f(\theta)| = M$

$$\Rightarrow |I_2| \le 2M \underbrace{\int_\varepsilon^{2\pi-\varepsilon} g_\rho(\theta)\,d\theta}_{\downarrow \text{ as } \rho\to 1 \atop 0}$$

Thus $|(f*g_\rho)(\phi) - f(\phi)| \to 0$ unif as $\rho\to 1$ ∎

Math 4200 partial fractions problem:

If $\frac{P(z)}{Q(z)}$ is a rational fcn with $\deg P < \deg Q$

and if $Q(z) = c(z-z_1)(z-z_2)\cdots(z-z_n)$

has n distinct roots

then Laplace transform theory yields a formula for the partial fractions decomposition of P/Q, as follows

$$\mathcal{L}^{-1}\left(\frac{P}{Q}\right)(t) = \sum_i \operatorname{res}\left(e^{zt}\frac{P(z)}{Q(z)}, z_i\right)$$

$$f(t) = \sum_i e^{z_i t}\frac{P(z_i)}{Q'(z_i)} \quad \text{since } z_i \text{ is a simple pole}$$

But $\mathcal{L}(f(t)) = \frac{P(z)}{Q(z)}$

$$\| $$

$$\sum_i \frac{P(z_i)}{Q'(z_i)}\frac{1}{z-z_i} = \sum_i \frac{A_i}{z-z_i} \quad \text{with } A_i = \frac{P(z_i)}{Q'(z_i)}$$

This is the partial fractions decomposition of $\frac{P(z)}{Q(z)}$ which you are used to finding via algebra

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$\boxed{\text{Homework exercise:}}$

The algorithm you are taught for repeated (linear) roots is

If $Q(z) = c(z-z_1)^{k_1}\cdots(z-z_\ell)^{k_\ell}$ $\quad z_1 \ldots z_\ell$ distinct, $k_1 + \cdots + k_\ell = \deg Q$

Then

$$\frac{P(z)}{Q(z)} = \sum_{i=1}^{\ell}\left(\sum_{j=1}^{k_i}\frac{A_{ij}}{(z-z_i)^j}\right)$$

- Use Laplace and inverse Laplace to show such a decomposition exists.
  (The difference from the "no-repeated-roots" case above is that you no longer have simple poles at the z_i, their order is k_i now.)
- Can you find a nice formula for the A_{ij}? (not required, but maybe fun?)