Math 4200
Wed 2 Nov.

<u>Residue Theorem</u> (general version)

Let $f: A \to \mathbb{C}$ analytic except at $\{z_1, \dots z_N\} \subset A$

$\gamma: I \to A$ closed curve (p.w. C')

<u>homotopic to a point in A</u>

Then $$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)\, I(\gamma; z_j)$$

<u>proof</u>: Expand f in Laur. series at each z_j:

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_j)^n}}_{S_j(z)} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_j)^n}_{A_j(z)} \qquad \text{in } D(z_j, r_j) \setminus \{z_j\}$$

$\uparrow$ converges to an analytic fcn in all of $\mathbb{C} \setminus \{z_j\}$

Let $g(z) := f(z) - \sum_{j=1}^{N} S_j(z)$. Then the singularities of g at z_j are removable, since $f(z) - S_j(z) = A_j(z)$ is analytic near z_j (and so is $-\sum_{\ell \neq j} S_\ell(z)$).

So g (extends to be) analytic in A

Since γ homotopic to a point in A,

$$\int_\gamma g(z)\,dz = 0$$

$$= \int_\gamma f(z) - \sum_{j=1}^{N} \int_\gamma S_j(z)\,dz$$

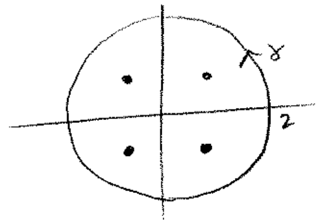
$$\int_\gamma S_j(z)\,dz = \int \sum_{n=1}^{\infty} \frac{b_n}{(z-z_j)^n}\,dz$$

$$\underbrace{\sum_{n=1}^{\infty} \int_\gamma \frac{b_n}{(z-z_j)^n}\,dz}_{= b_{-1} \int_\gamma \frac{1}{z-z_j}\,dz} = b_{-1}(2\pi i\, I(\gamma; z_j))$$

$$0 = \int_\gamma f(z)\,dz - 2\pi i \sum_{j=1}^{N} \mathrm{Res}(f; z_j)\, I(\gamma; z_j)$$

• example:

$$\oint_{|z|=2} \frac{z^2}{1+z^4}\,dz$$



$1 + z^4 = 0$

$z^4 = -1$

$e^{i\pi/4},\ e^{3i\pi/4},\ e^{-i\pi/4},\ e^{-3i\pi/4}$

Method 1

$f = \frac{g}{h}$ simple pole @ z_0

$\mathrm{res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}$

$$\mathrm{Res}(f, e^{i\pi/4}) = \frac{e^{2i\pi/4}}{4(e^{3i\pi/4})} = \frac{1}{4}e^{-i\pi/4}$$

$$\mathrm{Res}(f, e^{-i\pi/4}) = \frac{e^{-2i\pi/4}}{4e^{-3i\pi/4}} = \frac{1}{4}e^{i\pi/4}$$

$$\mathrm{Res}(f, e^{3i\pi/4}) = \frac{e^{6i\pi/4}}{4e^{9i\pi/4}} = \frac{1}{4}e^{-3i\pi/4}$$

$$\mathrm{Res}(f, e^{-3i\pi/4}) = \frac{e^{-6i\pi/4}}{4e^{-9i\pi/4}} = \frac{1}{4}e^{3i\pi/4}$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i\left(\sum \mathrm{res}\right) = 0.$$

Method 2 You can do substitution in contour integrals

$z = G(\zeta)$ G analytic, invertible on γ

$dz = G'(\zeta)\,d\zeta$

$$\int_\gamma f(z)\,dz = \int_{G^{-1}(\gamma)} f(G(\zeta))\,G'(\zeta)\,d\zeta$$

recall (?)

LHS $= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

$$\mathrm{RHS} = \int_a^b f\underbrace{\left(G(G^{-1}(\gamma(t)))\right)}_{f(\gamma(t))} G'(G^{-1}(\gamma(t)))\underbrace{\left(G^{-1}(\gamma(t))\right)'}_{(G^{-1})'(\gamma(t))\,\gamma'(t)}\,dt$$

chain rule for curves

subs $z = \frac{1}{\zeta}$, $\zeta = \frac{1}{z}$

If $\gamma(t) = 2e^{it}$

$\frac{1}{\gamma}(t) = \frac{1}{2}e^{-it}$

$$\oint_\gamma \frac{z^2}{1+z^4}\,dz = \int_{1/\gamma} \frac{\left(\frac{1}{\zeta}\right)^2}{1+\left(\frac{1}{\zeta}\right)^4}\left(-\frac{1}{\zeta^2}\right)d\zeta$$

$$= \oint_{|\zeta|=1/2} \frac{-1}{\zeta^4+1}\,d\zeta = \oint_{|\zeta|=1/2} \frac{1}{1+\zeta^4}\,d\zeta = 0 \quad \text{(no poles inside!)}$$

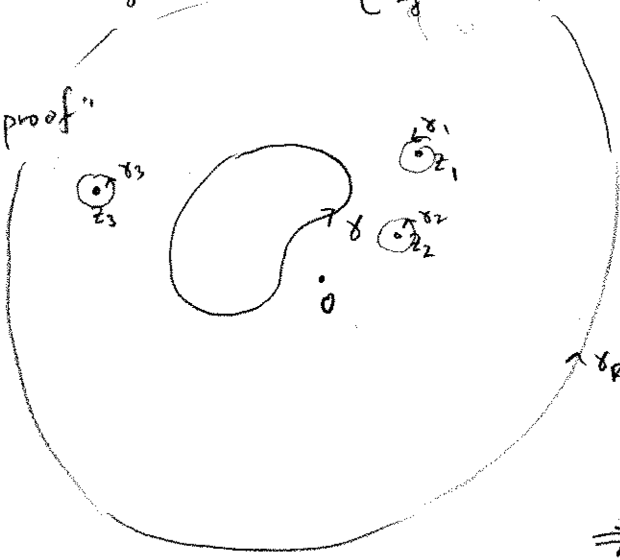
<u>Def</u> If f is analytic in a nbhd of ∞ (i.e. outside $D(0,R)$ some $R>0$),

then $\mathrm{Res}(f;\infty) := \mathrm{Res}\left(-\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right);0\right)$

<u>Thm</u>: Let γ be a simple closed curve in $\mathbb{C}$, traversed once counterclockwise.
Let f analytic along γ with only finitely many singularities outside γ
Then

$$\int_\gamma f(z)\,dz = -2\pi i \left\{ \sum_{z_j \text{ outside } \gamma} \mathrm{Res}(f;z_j) + \mathrm{Res}(f;\infty) \right\}$$

"proof"



$\gamma_R(t) = Re^{it}$

Let all $\{z_j\} \subset D(0,R)$.

Green's argument (Friday notes)

$$\Rightarrow \oint_{\gamma_R} f(z)\,dz - \oint_\gamma f(z)\,dz - \sum_j \oint_{\gamma_j} f(z)\,dz = 0$$

$$\Rightarrow \oint_\gamma f(z)\,dz = \underbrace{\oint_{\gamma_R} f(z)\,dz}_{} - 2\pi i \sum_j \mathrm{Res}(f,z_j)$$

$\zeta = \frac{1}{z}$

$z = Re^{it}$

$\zeta = \frac{1}{R} e^{-it}$

$$\oint_{|\zeta| = \frac{1}{R}} f\left(\frac{1}{\zeta}\right)\left(-\frac{1}{\zeta^2}\right) d\zeta$$

$= -2\pi i\,\mathrm{Res}(f,\infty)$

check those − signs!

in HW this applys to

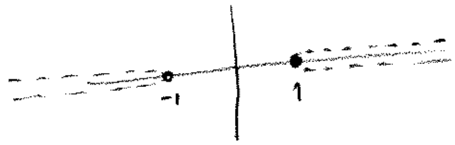
§ 4.3 #13,

possibly #19.

Notes on § 4.2 # 15.

$f(z) = \sqrt{z^2-1} = \sqrt{z-1}\sqrt{z+1}$.

Earlier, we found a natural simply connected domain of analyticity for f:



but could've also done $\quad f(z) = e^{\frac{1}{2}\left[\log(z-1) + \log(z+1)\right]}$

$0 < \arg(z-1) < 2\pi$
$0 < \arg(z+1) < 2\pi$

focus here

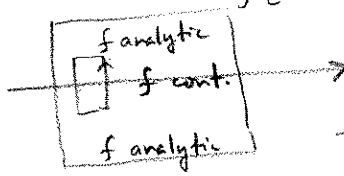
$\downarrow$

$\arg(z-1), \arg(z+1) \to 0$

$\arg(z-1), \arg(z+1) \to 2\pi$

$\frac{1}{2}[\arg(z-1) + \arg(z+1)] \to 2\pi$.

so f extends continuously from top and bottom along $\{z = x > 1\}$



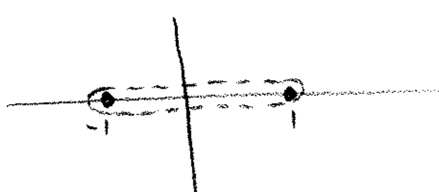
$\Rightarrow$ f analytic also along real axis

pf: Morera, i.e.
prove rectangle lemma
$\Rightarrow \exists F$ antiderivative
$\Rightarrow F' = f$ analytic.

Example 2.4.17
page 159.

so in fact
$f(z) = e^{\frac{1}{2}\left[\log(z-1) + \log(z+1)\right]}$

$0 \le \arg(z-1) \le 2\pi$
$0 \le \arg(z+1) \le 2\pi$ gives analytic fun in



#15. Compute $\displaystyle\oint_{|z|=2} \sqrt{z^2-1}\, dz$ for this f. Two ways possible

1. invert, $z = \frac{1}{3}$, i.e. residue at ∞
2. Laurent series for $\sqrt{z^2-1}$ valid for $|z| > 1$

$|w| < 1 \Rightarrow (1+w)^p = 1 + pw + \frac{p(p-1)}{2}w^2 + \dots \quad p = 2$

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